Concept lattices with negative information: a characterization theorem

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Abstract

Classical Formal Concept Analysis (FCA) extracts, represents and manages knowledge from positive information, i.e., its fundamental data model is a binary relation between a set of objects and attributes indicating the *presence* of a property in an object. However, some applications require to treat the *absence* of some property in an object as a negative information to be explicitly represented and managed, too. Although mixed (positive and negative) information has been addressed in the past in FCA, such approaches maintain the standard framework, which hides the specific semantics and avoids the further use of direct techniques and methods for mixed information. In this work, the foundations of FCA are extended and, in particular, mixed concept lattices are studied in depth. The main result of this work is a characterization theorem specifying in lattice-theoretic terms which lattices are isomorphic to a mixed concept lattice.

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1. Introduction

Formal concept analysis (FCA) constitutes a very successful mathematical approach to knowledge representation, with a rich theory as well as numerous

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practical applications. At the core of FCA is a basic incidence structure (called *formal context*) describing a set of objects and their attributes. From this, one can derive *formal concepts* which represent sets of objects with common attributes. Formal concepts come with a natural specialization/generalization order and the set of all formal concepts of a formal context can be shown to form a complete lattice with respect to that order. Conversely, it has been shown as part of the basic theorem of FCA [7] that every complete lattice is isomorphic to a concept lattice of some appropriately defined formal context.

The appropriate logical formalism for FCA is propositional Horn logic, i.e. implications between conjunctions of attributes. This logical representation is omnipresent in the areas of data mining and general knowledge discovery, and facilitates decision making in combination with pattern recognition, clustering, association and classification methods.

Per se, FCA is not well suited to handle negative information. Likewise, in most data mining and knowledge discovery frameworks, implications and association rules are typically built using positive information only. However, driven by requirements from practical situations, diverse attempts have been made to extend FCA to also represent negative information. For instance, Wille introduced different versions of negation (called *negation* and *opposition*) on the concept level [22] and characterized the structures thus obtained.

In this paper, we focus on the case where negation is available on the attribute level, i.e., we assume that for every (positive) attribute a we can also use an attribute $\overline{a}$ that holds exactly for those objects not having a. Note that the availability of such negative attributes in implications significantly enhances expressivity. For example, while it is easy to express with an implication over positive attributes that every father is a parent (through the implication $\mathit{father} \rightarrow \mathit{parent}$), this is not possible for the proposition that every parent is a father or mother. However, using negative attributes, this can be expressed by the implication $\overline{\mathit{father}}\ \overline{\mathit{mother}} \rightarrow \overline{\mathit{parent}}$. This shows that introducing negative attributes allows to express correspondences that are very natural and crucial for knowledge representation.

One way to formalize this setting (and reduce it to the classical FCA approach) is to use the apposition of the context and its negation [3, 8], i.e. we extend the original context by new attributes which represent the negated versions of the original attributes. However, as a basis for computations, this representation tends to be inefficient and redundant as observed by Missaoui et al. [16].

One of the reasons for this is that whenever negated attributes are available, the corresponding concept lattice exhibits certain additional properties. The central task that we tackle in this paper is to make this additional properties explicit by characterizing the lattices (which we call *mixed lattices*) that arise from negation-enhanced attributes. This requires to show two directions. First, we identify some properties that hold for all mixed lattices; showing that these properties indeed hold is not overly complicated. The other direction, however, requires us to show that every lattice satisfying the characterizing properties can indeed be obtained as mixed lattice of some formal context, which constitutes the more intricate part of our argument.

Our results provide a better understanding of the algebraic structures obtained in cases with mixed positive and negative information and will allow for designing better, specialized algorithms which exploit these specific structural properties.

The paper proceeds as follows: Section 2 will provide the necessary basics of FCA. Section 3 motivates and formally introduces the considered extension of "traditional" FCA with negated attributes by defining mixed derivation operators, mixed formal concepts and mixed attribute implications. Section 4 introduces the central notion of this paper: mixed lattices, and establishes some properties satisfied by them. Finally, Section 5 combines two of these properties into a characterization by showing that any lattice exhibiting these properties is isomorphic to the mixed lattice of some formal context.

2. Preliminaries

Assuming the reader to have primary knowledge about lattices [9], we recall some basics for a common understanding of notation Throughout the article we will consider bounded lattices and, as usual, supremum and infimum will be denoted by $\vee$ and $\wedge$ respectively, whereas $\top$ and $\perp$ will denote the maximum and the minimum elements. In a bounded lattice $\mathbb{L} = \langle L, \vee, \wedge, \top, \perp\rangle$, one element $j \in L$ is said to be join-irreducible or $\vee$-irreducible if $\ell_1 \vee \ell_2 = j$ implies $\ell_1 = j$ or $\ell_2 = j$ for all $\ell_1, \ell_2 \in L$. The set of $\vee$-irreducible elements in $\mathbb{L}$ are going to be denoted by $\mathtt{J}(\mathbb{L})$. The $\wedge$-irreducible elements are dually defined and $\mathtt{M}(\mathbb{L})$ denotes the set of $\wedge$-irreducible elements in $\mathbb{L}$. One element $a \in L$ is said to be an atom if $\ell \leq a$ implies $\ell = \perp$ or $\ell = a$ for all $\ell \in L$ and the set of atoms is denoted by $\mathtt{At}(\mathbb{L})$.

Definition 1 (atomistic lattice). *A lattice $\mathbb{L}$ is said to be atomistic if any $\vee$-irreducible element is an atom, i.e.* $\mathtt{J}(\mathbb{L}) = \mathtt{At}(\mathbb{L})$.

In addition, given one element $\ell \in L$, the up-set and the down-set of ℓ are $[\ell) = \{x \in L \mid \ell \leq x\}$ and $(\ell] = \{x \in L \mid x \leq \ell\}$ respectively. Finally, $\ell_{\mathtt{A}}$ denotes $(\ell] \cap \mathtt{At}(\mathbb{L})$ and $\ell^{\mathtt{M}}$ denotes $[\ell) \cap \mathtt{M}(\mathbb{L})$.

Now, we briefly present the basic notions related to Formal Concept Analysis (FCA) and attribute implications. See [7] for a much more detailed record. Basically, FCA encompasses a set of theoretical results, techniques and tools that allows one to extract knowledge from data, to reason using this knowledge and thereby to address practical problems. Information is usually given in terms of relations between data elements. Working with a large amount of data requires summarizing information in a way that the most relevant aspects are emphasized. Such a concise summary allowing one to work in an efficient way could be referred to as "knowledge" regarding the data. In FCA, such basic relational data is represented by formal contexts. Specifically, in the original formalization, a *formal context* is a triple $\mathbb{K} = \langle G, M, I\rangle$ where G and M are finite non-empty sets and $I \subseteq G \times M$ is a binary relation. The elements in G are called objects, the elements in M attributes and $\langle g, m\rangle \in I$ means that the object g has the attribute m. From this triple, two mappings ${}^{\uparrow}\colon 2^G \to 2^M$ and ${}^{\downarrow}\colon 2^M \to 2^G$, called derivation operators, are defined as follows:

$$X^{\uparrow} = \{m \in M \mid \langle g, m\rangle \in I \text{ for all } g \in X\} \tag{1}$$

$$Y^{\downarrow} = \{g \in G \mid \langle g, m\rangle \in I \text{ for all } m \in Y\} \tag{2}$$

The pair $({}^{\uparrow}, {}^{\downarrow})$ constitutes a Galois connection between $(2^G, \subseteq)$ and $(2^M, \subseteq)$ and, therefore, both compositions, ${}^{\uparrow}\circ{}^{\downarrow}$ and ${}^{\downarrow}\circ{}^{\uparrow}$, are closure operators whose closed sets constitute a lattice.

A first summary of (knowledge extracted from) the data stored in the formal context is given by these closed sets which lead to the notions of formal concept and concept lattice: A pair of subsets $\langle X, Y\rangle$ with $X \subseteq G$ and $Y \subseteq M$ such that $X^{\uparrow} = Y$ and $Y^{\downarrow} = X$ is called a *formal concept* where X is its *extent* and Y its *intent*. The set of all formal concepts with the relation

$$\langle X_1, Y_1\rangle \leq \langle X_2, Y_2\rangle \text{ if and only if } X_1 \subseteq X_2 \text{ (or equivalently, } Y_2 \subseteq Y_1) \tag{3}$$

is a lattice, which is called *concept lattice* and denoted by $\mathfrak{B}(\mathbb{K})$.

Focusing the attention on relationships between sets of attributes is a second way in which information can be summarized. These relationships among attribute sets are described in terms of *attribute implications* being expressions $A \to B$ where $A, B \subseteq M$. An implication $A \to B$ is said to *hold* in a context $\mathbb{K}$ if $A^{\downarrow} \subseteq B^{\downarrow}$, i.e. any object that has all the attributes in A has also all the attributes in B. In this situation, one also says that the formal context $\mathbb{K}$ is a *model* for (or: *satisfies*) $A \to B$, denoted by $\mathbb{K} \models A \to B$. A direct approach to represent this knowledge could be to consider all attribute implications that the context satisfies. This set is called *full implicational system*. Both knowledge extracted from the formal context, the full implicational system and the concept lattice, are indeed two views on the same knowledge. One advantage of the view given by attribute implications is that, although the cardinality of the full implicational system could be very large, fortunately, this set can be further reduced by using the well-known Armstrong Axioms. A set of implications Σ is considered a *complete implicational system for* $\mathbb{K}$ if it satisfies the following: an implication holds in $\mathbb{K}$ (i.e. it belongs to $\mathbb{K}$'s full implicational system) if and only if it can be inferred, by using Armstrong's Axioms, from Σ.

Observe that there exist different implicational systems for the same full implicational system. For a detailed study about several properties related with the size of these implicational systems, we refer to [21, 2]. Finally, there are several algorithms in the literature to calculate from a formal context both the concept lattice and an implicational system. The most cited one is the NextClosure Algorithm [4]. It produces the concept lattice and the so-called Duquenne-Guigues (or stem) basis [10] which is an implicational system whose cardinality (number of implications) is minimum.

3. Negative attributes

Full implicational systems obtained from formal contexts provide all the knowledge about the presence of properties in objects. However, some knowledge is not covered by implicational systems because they do not consider information relative to the absence of properties (attributes).

$\mathbb{K}$	a	b	c	d	e
o_1		$\times$	$\times$		$\times$
o_2	$\times$	$\times$			
o_3		$\times$	$\times$		$\times$
o_4			$\times$	$\times$	

Table 1: A formal context

For instance, from the formal context depicted in Table 1, the following Duquenne-Guigues basis is obtained: $\Sigma = \{e \to bc, d \to c, bc \to e, a \to b\}$. So, an attribute implication holds in the context if and only if it can be inferred from Σ by using Armstrong's axioms. Thus, for example, the implications $b \to d$ and $b \to c$ do not hold. Nevertheless, these two implications differ in how they don't hold in the context:

Case A: For the first implication ($b \to d$), any object that has the attribute b does not have the attribute d.

Case B: For the implication $b \to c$, the set of objects having attribute b, can be divided according to the presence or absence of the attribute c.

The only way to distinguish between these two situations is to enrich the expressivity of the language, so that *positive* and *negative* attributes are introduced to respectively represent the presence or absence of properties in objects. Thus, if $\overline{d}$ denotes the absence of attribute d (negative attribute), we can express that the implication $b \to \overline{d}$ holds in Table 1, corresponding with case A. On the other side, regarding case B, neither implication $b \to c$ nor $b \to \overline{c}$ holds. Therefore, such an extension of the implicational language allows to distinguish both cases.

It is natural that an increase in the language expressiveness requires to revisit the existing theoretical results, since often, a more general framework to deal with this kind of knowledge is needed. Some starting approaches in this line can be found in [16, 14, 15] and applications in [17, 11, 12]. In [18], we tackled this issue focusing on the problem of mining implications with positive and negative attributes from formal contexts. As a conclusion of that work, we emphasized the necessity of a full development of the algebraic framework, which is the aim of this paper. A first step in this direction was made in [19] and it is briefly presented in this section and at the beginning of Section 4.

First, we begin with the introduction of an extended notation that allows us to consider the negation of attributes. From now on, the set of attributes is denoted by M, and its elements by the letter m, possibly with subindices. That is, the lowercase character m is reserved for what we call *positive attributes*. We use $\overline{m}$ to denote the negation of the attribute m and $\overline{M}$ to denote the set $\{\overline{m} \mid m \in M\}$ whose elements will be called *negative attributes*.

Arbitrary elements in $M \cup \overline{M}$ are going to be denoted by the first letters in the alphabet: a, b, c, etc. and $\overline{a}$ denotes the opposite of a. That is, the symbol a could represent a positive or a negative attribute and, if $a = m \in M$ then $\overline{a} = \overline{m}$ and if $a = \overline{m} \in \overline{M}$ then $\overline{a} = m$.

Capital letters A, B, C,... denote subsets of $M \cup \overline{M}$. If $A \subseteq M \cup \overline{M}$, then $\overline{A}$ denotes the set of the opposite of attributes $\{\overline{a} \mid a \in A\}$ and the following sets are defined:

$$\mathrm{Pos}(A) = A \cap M, \ \mathrm{Neg}(A) = \overline{A} \cap M \text{ and } \mathrm{Tot}(A) = \mathrm{Pos}(A) \cup \mathrm{Neg}(A).$$

Note that $A = \mathrm{Pos}(A) \cup \overline{\mathrm{Neg}(A)}$.

Now, we extend the definitions of derivation operators, formal concepts and attribute implications to the case of mixed attributes.

Definition 2 (Mixed derivation operators). *Let $\mathbb{K} = \langle G, M, I\rangle$ be a formal context. We define the operators $\Uparrow: 2^G \to 2^{M\cup\overline{M}}$ and $\Downarrow: 2^{M\cup\overline{M}} \to 2^G$ as follows: for $X \subseteq G$ and $Y \subseteq M \cup \overline{M}$,*

$$\begin{aligned} X^{\Uparrow} = \ & \{m \in M \mid \langle g, m\rangle \in I \textit{ for all } g \in X\} \\ \cup \ & \{\overline{m} \in \overline{M} \mid \langle g, m\rangle \notin I \textit{ for all } g \in X\} \end{aligned} \tag{4}$$

$$\begin{aligned} Y^{\Downarrow} = \ & \{g \in G \mid \langle g, m\rangle \in I \textit{ for all } m \in Y\} \\ \cap \ & \{g \in G \mid \langle g, m\rangle \notin I \textit{ for all } \overline{m} \in Y\} \end{aligned} \tag{5}$$

Definition 3 (Mixed formal concept). *Let $\mathbb{K} = \langle G, M, I\rangle$ be a formal context. A* mixed formal concept *in $\mathbb{K}$ is a pair of subsets $\langle X, Y\rangle$ with $X \subseteq G$ and $Y \subseteq M \cup \overline{M}$ such $X^{\Uparrow} = Y$ and $Y^{\Downarrow} = X$.*

Definition 4 (Mixed attribute implication). *Let $\mathbb{K} = \langle G, M, I\rangle$ be a formal context and let $A, B \subseteq M \cup \overline{M}$. The context $\mathbb{K}$ satisfies a* mixed attribute implication *$A \to B$, denoted by $\mathbb{K} \models A \to B$, if $A^{\Downarrow} \subseteq B^{\Downarrow}$.*

For example, in Table 1, as we previously mentioned, two different situations were presented. Thus, in this new framework we have that $\mathbb{K} \not\models b \to d$ and $\mathbb{K} \models b \to \overline{d}$ whereas $\mathbb{K} \not\models b \to c$ but also $\mathbb{K} \not\models b \to \overline{c}$.

4. Mixed concept lattices

The goal of this paper is to develop the generalized lattice-theoretic framework for the mixed attribute case. In this section we are going to introduce the main results of this paper, providing the properties of the generalized concept lattice. The main pillar of our new framework are the two derivation operators introduced in Equations 4 and 5. The following theorem ensures that these two operators form a Galois connection:

Theorem 1. *Let $\mathbb{K} = \langle G, M, I \rangle$ be a formal context. The pair of derivation operators $(\Uparrow, \Downarrow)$ introduced in Definition 2 is a Galois connection.*

Proof. We need to prove that, for all subsets $X \subseteq G$ and $Y \subseteq M \cup \overline{M}$,

$$X \subseteq Y^{\Downarrow} \text{ if and only if } Y \subseteq X^{\Uparrow}$$

First, assume $X \subseteq Y^{\Downarrow}$. For all $a \in Y$, we distinguish two cases:

1. If $a \in \mathrm{Pos}(Y)$, there exists $m \in M$ with $a = m$ and, for all $g \in X$, since $X \subseteq Y^{\Downarrow}$, $\langle g, m \rangle \in I$ and therefore $a = m \in X^{\Uparrow}$.
2. If $a \in \mathrm{Neg}(Y)$, there exists $m \in M$ with $a = \overline{m}$ and, for all $g \in X$, since $X \subseteq Y^{\Downarrow}$, $\langle g, m \rangle \notin I$ and therefore $a = \overline{m} \in X^{\Uparrow}$.

Conversely, assume $Y \subseteq X^{\Uparrow}$ and $g \in X$. To ensure that $g \in Y^{\Downarrow}$, we need to prove that $\langle g, a \rangle \in I$ for all $a \in \mathrm{Pos}(Y)$ and $\langle g, a \rangle \notin I$ for all $a \in \mathrm{Neg}(Y)$, which is straightforward from $Y \subseteq X^{\Uparrow}$. $\square$

Therefore, the above theorem ensures that the two compositions, i.e. ${}^{\Uparrow \circ \Downarrow}$ and ${}^{\Downarrow \circ \Uparrow}$, are closure operators. Furthermore, as in the classical case, both closure operators provide two dually isomorphic lattices. We denote by $\mathfrak{B}^{\sharp}(\mathbb{K})$ the lattice of mixed concepts with the relation

$$\langle X_1, Y_1 \rangle \leq \langle X_2, Y_2 \rangle \text{ iff } X_1 \subseteq X_2 \text{ (or equivalently, iff } Y_1 \supseteq Y_2)$$

Moreover, as in standard FCA, mixed implications and mixed concept lattices make up the two sides of the same coin, i.e. the information mined from a mixed formal context may be dually represented by means of a set of mixed attribute implications or a mixed concept lattice.

As we shall see later in this section, unlike traditional concept lattices, mixed concept lattices are restricted to a specific lattice subclass. There exist specific

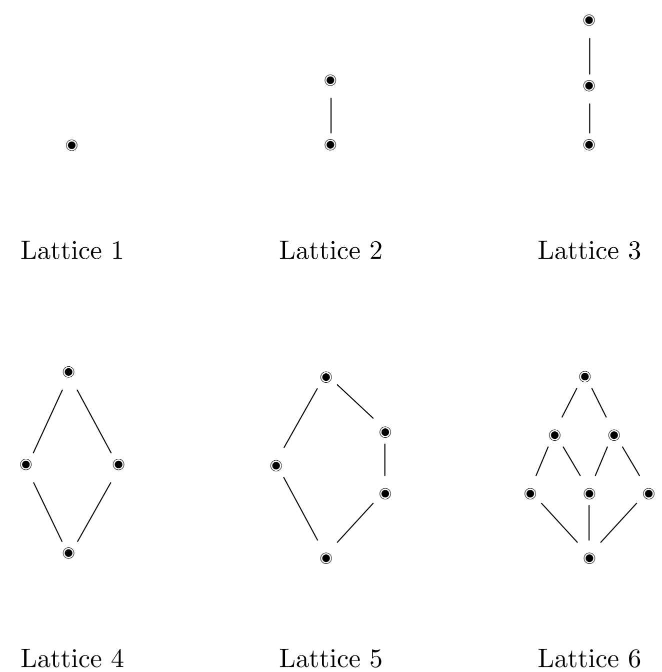

Table 2: Diagrams of some lattices

properties that lattices must satisfy to be considered a valid lattice structure which corresponds to a mixed formal context. In fact, this is the main goal of this paper: the characterization of those lattices which correspond to mixed concept lattices.

In the classical framework, every complete lattice is isomorphic to a concept lattice. However, in the mixed framework this property does not hold. For instance, in Table 2 six different lattices are depicted. In the original framework, all of them can be associated with concept lattices. Yet, as we shall prove later in this paper, lattices 3 and 5 cannot be associated with a mixed concept lattice.

The following two definitions characterize two kinds of significant sets of attributes that will be used later:

Definition 5 (Consistent sets). *Let* $\mathbb{K} = \langle G, M, I \rangle$ *be a formal context. A* *set* $A \subseteq M \cup \overline{M}$ *is called a* consistent set *if* $\mathrm{Pos}(A) \cap \mathrm{Neg}(A) = \varnothing$. *The set of consistent sets are going to be denoted by* $\mathbb{C}\mathrm{tts}$, *i.e.*

$$\mathbb{C}\mathrm{tts} = \{A \subseteq M \cup \overline{M} \mid \mathrm{Pos}(A) \cap \mathrm{Neg}(A) = \varnothing\}$$

If $A \in \mathbb{C}\mathrm{tts}$ then $|A| \leq |M|$ and, in the particular case where $|A| = |M|$, we have $\mathrm{Tot}(A) = M$. This situation induces the notion of *full* set:

Definition 6 (Full consistent sets). *Let* $\mathbb{K} = \langle G, M, I\rangle$ *be a formal context. A set* $A \subseteq M \cup \overline{M}$ *is said to be a* full consistent *set if* $A \in \mathbb{C}\mathrm{tts}$ *and* $\mathrm{Tot}(A) = M$.

In [11] the above full consistent notion was called a complete noncontradictory set. In [18, 19], we chose this new term to avoid confusion with other notions where the adjective "complete" is used: complete axiomatic system and complete implicational system.

In order to study the features that characterize mixed concept lattices, we introduce some lemmas that emphasize some of their specific properties. The first of these properties is the following: in the classical framework, for every context $\mathbb{K} = \langle G, M, I\rangle$, the concept lattice $\mathfrak{B}(\mathbb{K})$ is bounded by $\langle M^{\downarrow}, M\rangle$ and $\langle G, G^{\uparrow}\rangle$, where $M^{\downarrow}$ and $G^{\uparrow}$ can be arbitrary sets. However, in this mixed framework, the lower bound of $\mathfrak{B}^{\sharp}(\mathbb{K})$ is necessarily $\langle \varnothing, M \cup \overline{M}\rangle$.

Lemma 1. *Let* $\mathbb{K} = \langle G, M, I\rangle$ *be a formal context. Then* $\varnothing^{\Uparrow} = M \cup \overline{M}$, $\varnothing^{\Downarrow} = G$ *and* $(M \cup \overline{M})^{\Downarrow} = \varnothing$.

Remark 1. *From now on, we assume w.l.o.g. that* $\{g_1\}^{\Uparrow} = \{g_2\}^{\Uparrow}$ *implies* $g_1 = g_2$, *i.e. there is not two identical rows in the formal context.*

In the following lemma, we introduce other specific properties of the mixed concept lattices.

Lemma 2. *The following properties hold in any formal context* $\mathbb{K} = \langle G, M, I\rangle$:

1. $\{g\}^{\Uparrow}$ *is a full consistent set and* $\{g\}^{\Uparrow\Downarrow} = \{g\}$ *for all* $g \in G$.
2. $X^{\Uparrow} = \bigcap_{g \in X} \{g\}^{\Uparrow}$ *for all* $X \subseteq G$.

Proof.

1. $\{g\}^{\Uparrow}$ is a full consistent set because, for all $m \in M$, $\langle g, m\rangle \in I$ or $\langle g, m\rangle \notin I$ and $\{g\}^{\Uparrow} = \{m \in M \mid \langle g, m\rangle \in I\} \cup \{\overline{m} \in \overline{M} \mid \langle g, m\rangle \notin I\}$ being a disjoint union. Thus, $\mathrm{Tot}(\{g\}^{\Uparrow}) = M$ and $\mathrm{Pos}(\{g\}^{\Uparrow}) \cap \mathrm{Neg}(\{g\}^{\Uparrow}) = \varnothing$.
   On the other side, since $(\Uparrow, \Downarrow)$ is a Galois connection, $g_1 \in \{g\}^{\Uparrow\Downarrow}$ implies $\{g_1\}^{\Uparrow} \supseteq \{g\}^{\Uparrow\Downarrow\Uparrow} = \{g\}^{\Uparrow}$. Moreover, since both $\{g_1\}^{\Uparrow}$ and $\{g\}^{\Uparrow}$ are full consistent, one has that $\{g_1\}^{\Uparrow} = \{g\}^{\Uparrow}$ and, the assumption of no existence of repeated rows in the context ensures $g_1 = g$.
2. In the same way that occurs in the classical framework, since $(\Uparrow, \Downarrow)$ is a Galois connection between $(2^G, \subseteq)$ and $(2^{M\cup\overline{M}}, \subseteq)$, for any $X \subseteq G$, we have that $X^{\Uparrow} = \left(\bigcup_{g\in X}\{g\}\right)^{\Uparrow} = \bigcap_{g\in X}\{g\}^{\Uparrow}$. □

Another significant difference between the concept lattices and the mixed ones is related to the atom properties. In the standard framework, the inclusion $\mathtt{At}(\mathfrak{B}(\mathbb{K})) \subseteq \mathtt{J}(\mathfrak{B}(\mathbb{K})) \subseteq \{\langle\{g\}^{\uparrow\downarrow}, \{g\}^{\uparrow}\rangle \mid g \in G\}$ holds, but the equality is not necessarily fulfilled. The following theorem characterizes the atoms of the new concept lattice $\mathfrak{B}^{\sharp}(\mathbb{K})$, by means of the transformation of the above inclusion in an equality.

Theorem 2. *For every formal context $\mathbb{K} = \langle G, M, I\rangle$, its mixed concept lattice $\mathfrak{B}^{\sharp}(\mathbb{K})$ is atomistic and $\mathtt{At}(\mathfrak{B}^{\sharp}(\mathbb{K})) = \mathtt{J}(\mathfrak{B}^{\sharp}(\mathbb{K})) = \{\langle\{g\}^{\Uparrow\Downarrow}, \{g\}^{\Uparrow}\rangle \mid g \in G\}$.*

Proof. First, having fixed $g_0 \in G$, we are going to prove that the mixed concept $\langle\{g_0\}^{\Uparrow\Downarrow}, \{g_0\}^{\Uparrow}\rangle$ is an atom in $\mathfrak{B}^{\sharp}(\mathbb{K})$. If $\langle X, Y\rangle$ is a mixed concept such that $\langle\varnothing, M \cup \overline{M}\rangle < \langle X, Y\rangle \leq \langle\{g_0\}^{\Uparrow\Downarrow}, \{g_0\}^{\Uparrow}\rangle$, then $\{g_0\}^{\Uparrow} \subseteq Y = X^{\Uparrow} \subsetneq M \cup \overline{M}$. By Lemma 2, $\{g_0\}^{\Uparrow} \subseteq X^{\Uparrow} = \bigcap_{g\in X}\{g\}^{\Uparrow}$. Moreover, for all $g \in X \neq \varnothing$, by Lemma 2, both $\{g_0\}^{\Uparrow}$ and $\{g\}^{\Uparrow}$ are full consistent sets and, since $\{g_0\}^{\Uparrow} \subseteq \{g\}^{\Uparrow}$, we have $\{g_0\}^{\Uparrow} = \{g\}^{\Uparrow}$. Therefore, $\{g_0\}^{\Uparrow} = X^{\Uparrow} = Y$ and $\langle X, Y\rangle = \langle\{g_0\}^{\Uparrow\Downarrow}, \{g_0\}^{\Uparrow}\rangle$.

Conversely, if $\langle X, Y\rangle$ is an atom in $\mathfrak{B}^{\sharp}(\mathbb{K})$, then $X \neq \varnothing$ and there exists $g_0 \in X$. Since $(\Uparrow, \Downarrow)$ is a Galois connection, $\{g_0\}^{\Uparrow} \supseteq X^{\Uparrow} = Y$ and, therefore, $\langle\{g_0\}^{\Uparrow\Downarrow}, \{g_0\}^{\Uparrow}\rangle \leq \langle X, Y\rangle$. Finally, since $\langle X, Y\rangle$ is an atom, we have that $\langle X, Y\rangle = \langle\{g_0\}^{\Uparrow\Downarrow}, \{g_0\}^{\Uparrow}\rangle$. □

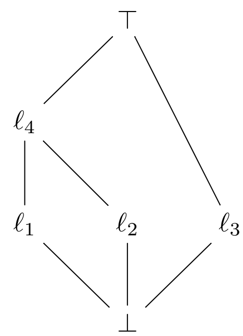

Figure 1: Atomistic lattice that is not isomorphic to any mixed concept lattice.

Observe that, with the assumption given in Remark 1, one has $\mathtt{At}(\mathfrak{B}^\sharp(\mathbb{K})) = \mathtt{J}(\mathfrak{B}^\sharp(\mathbb{K})) = \{\langle\{g\},\{g\}^\Uparrow\rangle \mid g \in G\}$ and, therefore, the number of atoms in a mixed concept lattice coincides with $|G|$. Moreover, for a mixed concept $\langle X, Y\rangle \in \mathfrak{B}^\sharp(\mathbb{K})$, the following two equivalences hold:

1. $\langle X, Y\rangle \in \mathtt{At}(\mathfrak{B}^\sharp(\mathbb{K}))$ if and only if Y is a full consistent set.
2. $\langle X, Y\rangle \neq \langle\varnothing, M \cup \overline{M}\rangle$ if and only if Y is a consistent set.

From the above result, we can deduce that being an atomistic lattice is a necessary condition for being isomorphic to a mixed concept lattice. However, the following example shows that it is not a sufficient condition.

Example 1. *The lattice $\mathbb{L}$ depicted in Figure 1 is atomistic, but it is not isomorphic to any mixed concept lattice. This assertion can be proven by reductio ad absurdum: assume that there exists a context $\mathbb{K} = \langle G, M, I\rangle$ and an isomorphism $f : \mathbb{L} \to \mathfrak{B}^\sharp(\mathbb{K})$. From Remark 1 and Theorem 2, we can assume that $G = \{g_1, g_2, g_3\}$ with $f(\ell_1) = \langle\{g_1\},\{g_1\}^\Uparrow\rangle$, $f(\ell_2) = \langle\{g_2\},\{g_2\}^\Uparrow\rangle$ and $f(\ell_3) = \langle\{g_3\},\{g_3\}^\Uparrow\rangle$. Since $\ell_1 \neq \ell_2$ and $\{g_1\}^\Uparrow$ and $\{g_2\}^\Uparrow$ are full consistent sets (see Lemma 2), there exists some $a \in M \cup \overline{M}$ such that $a \in \{g_1\}^\Uparrow$ and $\overline{a} \in \{g_2\}^\Uparrow$. In addition, since $\{g_3\}^\Uparrow$ is also a full consistent set, one has that $a \in \{g_3\}^\Uparrow$ or $\overline{a} \in \{g_3\}^\Uparrow$ and this disjunction is exclusive. Therefore $\{g_1\}^\Uparrow \cap \{g_3\}^\Uparrow \neq \{g_2\}^\Uparrow \cap \{g_3\}^\Uparrow$. This contradicts $\ell_1 \vee \ell_3 = \ell_2 \vee \ell_3 = \top$.*

On the way towards the characterization of mixed concept lattices, we present another milestone property.

Definition 7 (Atomic $\vee$-simplification law). *Let $\mathbb{L}$ be a lattice. $\mathbb{L}$ satisfies the atomic $\vee$-simplification law if, for all $a_1, a_2, a_3 \in \mathtt{At}(\mathbb{L})$, $a_1 \vee a_2 = a_1 \vee a_3$ implies $a_2 = a_3$.*

The above law resembles the well-known simplification law from Group Theory, in this case limited to atomic elements. Notice that if a lattice satisfies the $\vee$-simplification law, i.e. for all $\ell_1, \ell_2, \ell_3 \in L$, $\ell_1 \vee \ell_2 = \ell_1 \vee \ell_3$ implies $\ell_2 = \ell_3$, then the lattice has at most two elements.

Theorem 3. *$\mathfrak{B}^\sharp(\mathbb{K})$ satisfies the atomic $\vee$-simplification law for any formal context $\mathbb{K}$.*

Proof. Consider three atoms $a_1 = \langle \{g_1\}^{\Uparrow\Downarrow},\ \{g_1\}^{\Uparrow}\rangle$, $a_2 = \langle \{g_2\}^{\Uparrow\Downarrow},\ \{g_2\}^{\Uparrow}\rangle$, $a_3 = \langle \{g_3\}^{\Uparrow\Downarrow}, \{g_3\}^{\Uparrow}\rangle \in \mathtt{At}(\mathfrak{B}^\sharp(\mathbb{K}))$. By item 1 in Lemma 2, $\{g_1\}^{\Uparrow}, \{g_2\}^{\Uparrow}$ and $\{g_3\}^{\Uparrow}$ are full consistent sets. We are going to prove that $\langle \{g_2\}^{\Uparrow\Downarrow}, \{g_2\}^{\Uparrow}\rangle \neq \langle \{g_3\}^{\Uparrow\Downarrow}, \{g_3\}^{\Uparrow}\rangle$, implies $\langle \{g_1\}^{\Uparrow\Downarrow}, \{g_1\}^{\Uparrow}\rangle \vee \langle \{g_2\}^{\Uparrow\Downarrow}, \{g_2\}^{\Uparrow}\rangle \neq \langle \{g_1\}^{\Uparrow\Downarrow}, \{g_1\}^{\Uparrow}\rangle \vee \langle \{g_3\}^{\Uparrow\Downarrow}, \{g_3\}^{\Uparrow}\rangle$.

In effect, if $\{g_2\}^{\Uparrow} \neq \{g_3\}^{\Uparrow}$ then there exists $m \in \{g_2\}^{\Uparrow}$ such that $\overline{m} \in \{g_3\}^{\Uparrow}$.

- If $m \in \{g_1\}^{\Uparrow}$ then $m \in \{g_1\}^{\Uparrow} \cap \{g_2\}^{\Uparrow}$, $m \notin \{g_1\}^{\Uparrow} \cap \{g_3\}^{\Uparrow}$ and, therefore $a_1 \vee a_2 \neq a_1 \vee a_3$
- If $\overline{m} \in \{g_1\}^{\Uparrow}$ then $\overline{m} \notin \{g_1\}^{\Uparrow} \cap \{g_2\}^{\Uparrow}$, $\overline{m} \in \{g_1\}^{\Uparrow} \cap \{g_3\}^{\Uparrow}$ and, therefore $a_1 \vee a_2 \neq a_1 \vee a_3$

We have a contradiction and then $a_2 = a_3$. $\square$

The following example shows how the properties introduced up to now are necessary but still not sufficient conditions to characterize the mixed concept lattices.

Example 2. *The lattice depicted in Figure 2 is atomistic and fulfills the atomic $\vee$-simplification law presented above, but it is not isomorphic to any mixed concept lattice. This last assertion can be proved by reductio ad absurdum: assume that there exists a context $\mathbb{K} = \langle G, M, I\rangle$ and an isomorphism $f : \mathbb{L} \to \mathfrak{B}^\sharp(\mathbb{K})$.*

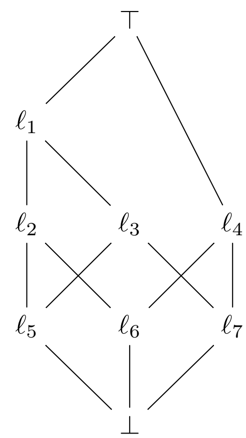

Figure 2: Example of lattice that is not isomorphic to any mixed concept lattice

From Remark 1 and Theorem 2, we can assume that $G = \{g_1, g_2, g_3\}$ *with* $f(\ell_5) = \langle\{g_1\}, \{g_1\}^{\Uparrow}\rangle$, $f(\ell_6) = \langle\{g_2\}, \{g_2\}^{\Uparrow}\rangle$ *and* $f(\ell_7) = \langle\{g_3\}, \{g_3\}^{\Uparrow}\rangle$. *Therefore,* $f(\ell_1) = \langle G, G^{\Uparrow}\rangle = f(\top)$, *which is a contradiction.*

The above example shows that we have to continue the search for sufficient conditions for characterizing mixed concept lattices.

Definition 8 (Opposite elements). *Let* $\mathbb{L}$ *be a finite lattice. For each* $\ell \in L$, *we define its* opposite element *as follows:*

$$\ell^{op} = \bigvee\{x \in L \mid \ell \wedge x = \perp\}$$

Observe that the opposite of a $\wedge$-irreducible element is not necessarily $\wedge$-irreducible (see for example the eight-element Boolean Algebra). Now, we are introducing some properties related to opposite elements:

Lemma 3. *Let* $\mathbb{L}$ *be an arbitrary finite lattice. For all* $\ell \in L$,

$$\ell^{op} \geq \bigvee\{x \in \mathtt{At}(\mathbb{L}) \mid x \notin \ell_{\mathtt{A}}\}$$

In addition, if $\mathbb{L}$ *is atomistic, the equality* $\ell^{op} = \bigvee\{x \in \mathtt{At}(\mathbb{L}) \mid x \notin \ell_{\mathtt{A}}\}$ *holds.*

Proof. The inequality $\ell^{op} \geq \bigvee\{x \in \mathtt{At}(\mathbb{L}) \mid x \notin \ell_{\mathtt{A}}\}$ is immediate because $x \notin \ell_{\mathtt{A}}$ implies $\ell \wedge x = \perp$, for all $x \in \mathtt{At}(\mathbb{L})$.

We assume now that $\mathbb{L}$ is atomistic and prove the contrary inequality. To ensure that $\ell^{op} \leq \bigvee\{x \in \mathtt{At}(\mathbb{L}) \mid x \notin \ell_{\mathtt{A}}\}$, by Definition 8, it is sufficient to prove that, for all $x \in L$, $\ell \wedge x = \perp$ implies $x \leq \bigvee\{x \in \mathtt{At}(\mathbb{L}) \mid x \notin \ell_{\mathtt{A}}\}$. In effect, if $\ell \wedge x = \perp$ then $x_{\mathtt{A}} \cap \ell_{\mathtt{A}} = \varnothing$. Therefore, $x_{\mathtt{A}} \subseteq \{x \in \mathtt{At}(\mathbb{L}) \mid x \notin \ell_{\mathtt{A}}\}$ and, since $\mathbb{L}$ is atomistic, $x = \bigvee x_{\mathtt{A}} \leq \bigvee\{x \in \mathtt{At}(\mathbb{L}) \mid x \notin \ell_{\mathtt{A}}\}$. □

Definition 9 ($\wedge$-complemented lattice). *A lattice $\mathbb{L}$ is said to be $\wedge$-complemented if for all $\ell \in \mathtt{M}(\mathbb{L})$ we have that ℓ^{op} is a complement of ℓ, i.e. $\ell^{op} \wedge \ell = \perp$ and $\ell^{op} \vee \ell = \top$.*

The following lemma, which is straightforward, establishes the connection between opposite elements and complements.

Lemma 4. *Let $\mathbb{L}$ be a $\wedge$-complemented lattice and $\ell \in \mathtt{M}(\mathbb{L})$. Then, any complement of ℓ belongs to $(\ell^{op}]$.*

Theorem 4. *Any mixed concept lattice is $\wedge$-complemented.*

Proof. Let $\mathbb{K} = \langle G, M, I\rangle$ be a formal context. By Theorem 2, the mixed concept lattice $\mathfrak{B}^\sharp(\mathbb{K})$ is atomistic. Given a $\wedge$-irreducible element $\langle A_1, B_1\rangle \in \mathfrak{B}^\sharp(\mathbb{K})$, by Lemma 3, we have that $\langle A_1, B_1\rangle^{op} = \vee\{\langle g^{\Uparrow\Downarrow}, g^{\Uparrow}\rangle | g \in G, B_1 \nsubseteq g^{\Uparrow}\}$ and, therefore $\langle A_1, B_1\rangle \vee \langle A_1, B_1\rangle^{op} = \langle G, G^{\Uparrow}\rangle$.

To conclude this proof we prove that $\langle A_1, B_1\rangle \wedge \langle A_1, B_1\rangle^{op} = \langle\emptyset, M \cup \overline{M}\rangle$. First, we are going to prove that there exists $a \in B_1$ such that $\overline{a} \in B_2$ where $\langle A_2, B_2\rangle = \langle A_1, B_1\rangle^{op}$. Because, in this situation, $a, \overline{a} \in \langle B_1 \cup B_2\rangle^{\Downarrow\Uparrow}$ and therefore $B_1 \cup B_2 = M \cup \overline{M}$.

Since $\langle A_1, B_1\rangle$ is $\wedge$-irreducible, there exists a unique concept $\langle A_3, B_3\rangle$ covering [1] $\langle A_1, B_1\rangle$. $\langle A_3, B_3\rangle > \langle A_1, B_1\rangle$ implies that there exists some $a \in B_1 \smallsetminus B_3$.

[1]In a poset $\langle P, \leq\rangle$, we say that an element $y \in P$ covers another element $x \in P$ if $x < y$ and there is no $z \in P$ such that $x < z < y$. In addition, when x is $\wedge$-irreducible, y covers x when, for all element $z \in P$, $x < z$ iff $y \leq z$.

For all $g \in G$ we prove that $a \in g^{\Uparrow}$ implies $\langle g^{\Uparrow\Downarrow}, g^{\Uparrow}\rangle \leq \langle A_1, B_1\rangle$. By reductio ad absurdum, let g_1 such that $a \in g_1^{\Uparrow}$ and $\langle g_1^{\Uparrow\Downarrow}, g_1^{\Uparrow}\rangle \nleq \langle A_1, B_1\rangle$. We have that $\langle g_1^{\Uparrow\Downarrow}, g_1^{\Uparrow}\rangle \vee \langle A_1, B_1\rangle > \langle A_1, B_1\rangle$ and therefore $\langle g_1^{\Uparrow\Downarrow}, g_1^{\Uparrow}\rangle \vee \langle A_1, B_1\rangle \geq \langle A_3, B_3\rangle$ and $a \in B_3$ which renders a contradiction.

Since $\langle A_2, B_2\rangle = \langle A_1, B_1\rangle^{op} = \vee\{\langle g^{\Uparrow\Downarrow}, g^{\Uparrow}\rangle | g \in G, B_1 \nsubseteq g^{\Uparrow}\}$ we have that $\overline{a} \in B_2$ because $a \notin g^{\Uparrow}$ for all g such that $B_1 \nsubseteq g^{\Uparrow}$ and $g^{\Uparrow}$ is full consistent. $\square$

5. Characterizing Mixed Concept Lattices.

In the previous section we proved that, for every context $\mathbb{K}$, the mixed concept lattice $\mathfrak{B}^{\sharp}(\mathbb{K})$ has the following properties:

(C1) It is atomistic.

(C2) The atomic $\vee$-simplification law holds.

(C3) It is $\wedge$-complemented.

Therefore, these properties establish necessary conditions for an arbitrary lattice to be isomorphic to a mixed concepts lattice.

In this section, we want to prove that these properties together are also sufficient conditions, providing a characterization theorem. Also, we will investigate mutual relationships between the conditions. In fact, there exists a strong connection between Conditions (C2) and (C3): any lattice which is atomistic and $\wedge$-complemented satisfies the atomic $\vee$-simplification law.

Theorem 5. *Let $\mathbb{L}$ be an atomistic lattice. If $\mathbb{L}$ is $\wedge$-complemented, then the atomic $\vee$-simplification law holds.*

Proof. Let $a_1, a_2, a_3 \in \mathtt{At}(\mathbb{L})$ where $a_1 \vee a_2 = a_1 \vee a_3$. Trivially, if $a_1 = a_2$ or $a_1 = a_3$, then $a_1 = a_2 = a_3$. Consider that $a_1 \neq a_2$ and $a_1 \neq a_3$. It is straightforward that $(a_1 \vee a_2)^{\mathtt{M}} = a_1^{\mathtt{M}} \cap a_2^{\mathtt{M}} = (a_1 \vee a_3)^{\mathtt{M}} = a_1^{\mathtt{M}} \cap a_3^{\mathtt{M}}$.

To prove that $a_2^{\mathtt{M}} = a_3^{\mathtt{M}}$ (i.e. $a_2 = a_3$) it is sufficient to prove that for all $m \in a_2^{\mathtt{M}} \smallsetminus a_1^{\mathtt{M}}$, we have that $m \in a_3^{\mathtt{M}} \smallsetminus a_1^{\mathtt{M}}$ and vice versa.

Consider $m \in a_2^{\mathtt{M}} \smallsetminus a_1^{\mathtt{M}}$ and suppose that $m \notin a_3^{\mathtt{M}}$. Since $a_3 \nleq m$ by Lemma 3, $a_3 \leq m^{op}$ and, similarly, $a_1 \leq m^{op}$. Thus, we deduce $a_1 \vee a_3 \leq m^{op}$. We have that $a_2 \leq a_1 \vee a_2 = a_1 \vee a_3 \leq m^{op}$, so $a_2 \leq m \wedge m^{op} = \perp$ and there is a contradiction because $a_2 \in \mathtt{At}(\mathbb{L})$.

The other inclusion can be proved analogously. $\square$

The reciprocal implication of the above theorem does not hold, as the following example shows.

Example 3. *The lattice depicted in Figure 2 is atomistic and satisfies the atomic $\vee$-simplification law, whereas it is not $\wedge$-complemented.*

In addition, Conditions (C1) and (C3) are independent. The previous example is an atomistic lattice which is not $\wedge$-complemented. On the other side, the pentagon (see Lattice 5 in Table 2) is $\wedge$-complemented but is not atomistic.

The following theorem is the main result of this paper, providing a characterization of those lattices that correspond to mixed concept lattices obtained from a formal context.

Theorem 6. *A finite lattice $\mathbb{L}$ is $\wedge$-complemented and atomistic if and only if there exists a context $\mathbb{K}$ such that $\mathbb{L}$ is isomorphic to $\mathfrak{B}^\sharp(\mathbb{K})$, $\mathbb{L} \cong \mathfrak{B}^\sharp(\mathbb{K})$.*

Proof. Given an atomistic $\wedge$-complemented lattice $\mathbb{L}$, consider the context $\mathbb{K}(\mathbb{L}) = \langle \mathtt{At}(\mathbb{L}), \mathtt{M}(\mathbb{L}), \leq \rangle$ and the following mapping:

$$h\colon \mathbb{L} \to \mathfrak{B}^\sharp(\mathbb{K}(\mathbb{L})) \text{ with } h(\ell) = \langle \ell_{\mathtt{A}}, \ell^{\mathtt{M}} \cup \ell^{\mathtt{M}}_{op} \rangle$$

where $\ell^{\mathtt{M}}_{op} = \{\overline{m} \mid m \in \mathtt{M}(\mathbb{L}), \ell \leq m^{op}\}$.

First, we are going to prove that the mapping h is well defined, i.e. $h(\ell)$ is a concept in the lattice $B^\sharp(\mathbb{K}(\mathbb{L}))$:

- To prove ${\ell_{\mathtt{A}}}^{\Uparrow} = \ell^{\mathtt{M}} \cup \ell^{\mathtt{M}}_{op}$, it is sufficient to check the following equalities:
  - $\ell^{\mathtt{M}} = \{m \in \mathtt{M}(\mathbb{L}) \mid a \leq m, \forall a \in \ell_{\mathtt{A}}\}$. Due to the definition of $\ell^{\mathtt{M}}$ and $\ell_{\mathtt{A}}$, and transitivity of $\leq$, the inclusion $\ell^{\mathtt{M}} \subseteq \{m \in \mathtt{M}(\mathbb{L}) \mid a \leq m, \forall a \in \ell_{\mathtt{A}}\}$ is straightforward. Conversely, if m is a meet-irreducible element with $a \leq m$ for all $a \in \ell_{\mathtt{A}}$, since $\mathbb{L}$ is atomistic, $\ell = \bigvee_{a \in \ell_{\mathtt{A}}} a \leq m$ and $m \in \ell^{\mathtt{M}}$.

- $\ell^{\mathtt{M}}_{op} = \{\overline{m} \mid m \in \mathtt{M}(\mathbb{L}), a \not\leq m, \forall a \in \ell_{\mathtt{A}}\}$. For $m \in \mathtt{M}(\mathbb{L})$, we are going to prove that $\ell \leq m^{op}$ if and only if $a \not\leq m, \forall a \in \ell_{\mathtt{A}}$. If $\ell \leq m^{op}$ and $a \in \ell_{\mathtt{A}}$, then a is an atom of $\mathbb{L}$ with $a \leq \ell \leq m^{op}$. Since $\mathbb{L}$ is $\wedge$-complemented, $m \wedge m^{op} = \bot$ and, therefore $a \not\leq m$. Conversely, if $m \in \mathtt{M}(\mathbb{L})$ with $a \not\leq m$ for all $a \in \ell_{\mathtt{A}}$ then $\ell_{\mathtt{A}} \subseteq \mathtt{At}(\mathbb{L}) \smallsetminus m_{\mathtt{A}}$ and, since $\mathbb{L}$ is atomistic, by Lemma 3, $\ell = \bigvee_{a \in \ell_{\mathtt{A}}} a \leq \bigvee_{a \in \mathtt{At}(\mathbb{L}) \smallsetminus m_{\mathtt{A}}} a = m^{op}$

- On the other side, we prove that $\ell_{\mathtt{A}} = \left(\ell^{\mathtt{M}} \cup \ell^{\mathtt{M}}_{op}\right)^{\Downarrow}$. That is, any atom a satisfies $a \leq \ell$ if and only if $a \leq m$ for all $m \in \ell^{\mathtt{M}}$ and $a \not\leq m$ for all $m \in \mathtt{M}(\mathbb{L})$ with $\ell \leq m^{op}$.

  If $a \leq \ell$, by transitivity, $a \leq m$ for all $m \in \ell^{\mathtt{M}}$ and, for all $m \in \mathtt{M}(\mathbb{L})$ with $\ell \leq m^{op}$, since $\mathbb{L}$ is $\wedge$-complemented, $a \not\leq m$. Conversely, if $a \leq m$ for all $m \in \ell^{\mathtt{M}}$ then $a \leq \bigwedge \ell^{\mathtt{M}} = \ell$.

Once h has been proved to be well-defined, we are going to prove that h is a lattice homomorphism:

- It is straightforward that $h(\ell_1) \wedge h(\ell_2) = h(\ell_1 \wedge \ell_2)$, for all $\ell_1, \ell_2 \in L$, because its extents are $\ell_{1\mathtt{A}} \cap \ell_{2\mathtt{A}} = (\ell_1 \wedge \ell_2)_{\mathtt{A}}$.

- To prove that $h(\ell_1) \vee f(\ell_2) = h(\ell_1 \vee \ell_2)$, for all $\ell_1, \ell_2 \in L$, we focus on the coincidence of its intents.

$$
\begin{aligned}
(\ell_1^{\mathtt{M}} \cup \ell_{1}{}^{\mathtt{M}}_{op}) \cap (\ell_2^{\mathtt{M}} \cup \ell_{2}{}^{\mathtt{M}}_{op}) &= (\ell_1^{\mathtt{M}} \cap \ell_2^{\mathtt{M}}) \cup (\ell_1^{\mathtt{M}} \cap \ell_{2}{}^{\mathtt{M}}_{op}) \cup (\ell_{1}{}^{\mathtt{M}}_{op} \cap \ell_2{}^{\mathtt{M}}) \cup (\ell_{1}{}^{\mathtt{M}}_{op} \cap \ell_{2}{}^{\mathtt{M}}_{op}) \\
&= (\ell_1^{\mathtt{M}} \cap \ell_2^{\mathtt{M}}) \cup (\ell_{1}{}^{\mathtt{M}}_{op} \cap \ell_{2}{}^{\mathtt{M}}_{op}) = (\ell_1 \vee \ell_2)^{\mathtt{M}} \cup (\ell_1 \vee \ell_2)^{\mathtt{M}}_{op}
\end{aligned}
$$

Finally, we prove that h is a bijection. Since the lattice is atomistic, if $h(\ell_1) = h(\ell_2)$, then their extents coincide and $\ell_1 = \bigvee \ell_{1\mathtt{A}} = \bigvee \ell_{2\mathtt{A}} = \ell_2$. Therefore, h is injective. To prove that h is surjective, consider a concept $\langle X, Y\rangle \in \mathfrak{B}^\sharp(\mathbb{K}(\mathbb{L}))$ and the element $\ell = \bigvee X \in L$. Since $X \subseteq \ell_{\mathtt{A}}$, we have that $\langle X, Y\rangle \leq h(\ell)$. Contrariwise, $h(\ell) \leq \langle X, Y\rangle$ because $Y \subseteq \ell^{\mathtt{M}} \cup \ell^{\mathtt{M}}_{op}$: Considering $m \in \mathtt{M}(\mathbb{L})$, we distinguish two cases:

- If $m \in Y = X^{\Uparrow}$ then $x \leq m$ for all $x \in X$ and, therefore, $\ell \leq m$ i.e. $m \in \ell^{\mathtt{M}}$.

- If $\overline{m} \in Y$, then $x \nleq m$ for all $x \in X$. Therefore $X \subseteq \mathtt{At}(\mathbb{L}) \smallsetminus m_{\mathtt{A}}$ and, by Lemma 3, $\ell = \bigvee X \leq \bigvee\{x \in \mathtt{At}(\mathbb{L}) \mid x \notin m_{\mathtt{A}}\} = m^{op}$. That is, $\overline{m} \in \ell^{\mathtt{M}}_{op}$. $\square$

To conclude this section, we illustrate how the strong connection between concept lattices and formal contexts in the classical framework has its counterpart in the mixed framework. In particular, we illustrate the way in which, from an atomistic $\wedge$-complemented lattice, a formal context can be built such that its mixed concept lattice is isomorphic to the initial lattice.

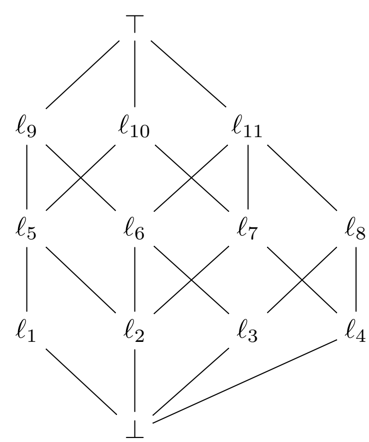

Figure 3: Example of an atomistic $\wedge$-complemented lattice.

Example 4. *Let us consider the lattice* $\mathbb{L}$ *depicted in Figure 3. This lattice is atomistic (i.e.* $\mathtt{J}(\mathbb{L}) = \mathtt{At}(\mathbb{L}) = \{\ell_1, \ell_2, \ell_3, \ell_4\}$*) and* $\wedge$*-complemented because its* $\wedge$*-irreducible elements are* ℓ_1*,* ℓ_8*,* ℓ_9*,* ℓ_{10} *and* ℓ_{11}*, and their opposite elements, which are also complement, are* ℓ_{11}*,* ℓ_5*,* ℓ_4*,* ℓ_3 *and* ℓ_1*, respectively.*

Table 3 shows the context $\mathbb{K}(\mathbb{L}) = \langle \mathtt{At}(\mathbb{L}), \mathtt{M}(\mathbb{L}), \leq \rangle$ *and it is just a matter of computation to check that* $\mathfrak{B}^\sharp(\mathbb{K}(\mathbb{L}))$ *is isomorphic to* $\mathbb{L}$*.*

6. The negation issue: a comparison of related approaches

Although FCA is underpinned on positive information, driven by the practical needs and the applications, diverse attempts have been made to extend

$\mathbb{K}(\mathbb{L})$	ℓ_1	ℓ_8	ℓ_9	ℓ_{10}	ℓ_{11}
ℓ_1	$\times$		$\times$	$\times$	
ℓ_2			$\times$	$\times$	$\times$
ℓ_3		$\times$	$\times$		$\times$
ℓ_4		$\times$		$\times$	$\times$

Table 3: The formal context $\mathbb{K}(\mathbb{L}) = \langle \mathtt{At}(\mathbb{L}), \mathtt{M}(\mathbb{L}), \leq \rangle$

FCA to also represent negative information. It can be made at two levels: by considering negation of attributes or concepts.

In this paper, the negation of the attributes has been taken under consideration. On the same level, Ganter and Wille [7, Page 60] suggested how to *dichotomize* a context. A context $\mathbb{K} = \langle G, M, I \rangle$ is dichotomous if, for all $m_1 \in M$ there exists $m_2 \in M$ such that $\langle g, m_1 \rangle \in I$ iff $\langle g, m_2 \rangle \notin I$ for all object $g \in G$. For every dichotomous context $\mathbb{K}$, $\mathfrak{B}^\sharp(\mathbb{K}) \cong \mathfrak{B}(\mathbb{K})$ holds and, therefore, the characterization given in this paper (Theorem 6) applies. There exists a significant number of articles in the literature following this line. For instance, in [16], Missaoui, Nourine and Renaud considered the apposition of the context and its negation, i.e. $\mathbb{K}|\overline{\mathbb{K}} = \langle G, M \cup \overline{M}, I \cup \overline{I} \rangle$ where $\langle g, \overline{m} \rangle \in \overline{I}$ iff $\langle g, m \rangle \notin I$. It is straightforward that $\mathbb{K}|\overline{\mathbb{K}}$ is dichotomous and $\mathfrak{B}^\sharp(\mathbb{K}) = \mathfrak{B}(\mathbb{K}|\overline{\mathbb{K}})$. An alternative view was given in [11] where, applying FCA to Machine Learning, Kuznetsov used k-dimensional dichotomic scales, as defined in [7, Page 57]. Context are dichotomized by using the scale $\mathbb{D}_M = \bowtie_{m \in M} \mathbb{D}_m$, which is the semiproduct[2] of the dichotomic scales $\mathbb{D}_m = \langle \{0,1\}, \{m, \overline{m}\}, \{\langle 1, m \rangle, \langle 0, \overline{m} \rangle\} \rangle$. The scale $\mathbb{D}_M$ is dichotomous, it has $2^{|M|}$ objects and $2|M|$ attributes, and its (mixed) concept lattice has $3^{|M|} + 1$ formal concepts.

Among the approaches considering negation on the concept level, Wille [22] introduced different versions of negation (called *weak negation* and *weak op-*

[2]The semiproduct of two contexts $\mathbb{K}_1 = \langle G_1, M_1, I_1 \rangle$ and $\mathbb{K}_2 = \langle G_2, M_2, I_2 \rangle$ is defined in [7, Page 46] as $\mathbb{K}_1 \bowtie \mathbb{K}_2 = \langle G_1 \times G_2, (\{1\} \times M_1) \cup (\{2\} \times M_2), \bigtriangledown \rangle$ where $(g_1, g_2) \bigtriangledown (j, m)$ iff $g_i I_i m$ for $i \in \{1, 2\}$.

position) and characterized the structures thus obtained. In a formal context $\mathbb{K} = \langle G, M, I\rangle$, the weak negation of a concept $\langle A, B\rangle$ is defined as $\langle A, B\rangle^{\triangle} = \langle (G \setminus A)^{\uparrow\downarrow}, (G \setminus A)^{\uparrow}\rangle$ whereas the weak opposition is defined as $\langle A, B\rangle^{\triangledown} = \langle (M \setminus B)^{\downarrow}, (M \setminus B)^{\downarrow\uparrow}\rangle$. A concept lattice equipped with these two operations is called a *concept algebra*. Wille also introduced the notion of *weakly dicomplemented lattice* to capture the equational theory of concept algebras. Several original laws are expected to be fulfilled when a negation is introduced, namely *principium exclusi tertii*, *principium contradictionis* and *duplex negation affirmat*. However, they do not necessarily hold in a weak negation or in a weak opposition. In [13], Kwuida, Tepavčević and Šešelja studied the subsets of concept algebras where those laws can be ensured. Ganter and Kwuida stated that finite distributive concept algebras are exactly finite distributive weakly dicomplemented lattices [5].

It is straightforward that, in a mixed concept lattice $\mathfrak{B}^{\sharp}(\mathbb{K})$, the weak negation of a concept coincides with our notion of opposite element, i.e. $\langle A, B\rangle^{\triangle} = \langle A, B\rangle^{op}$ (see Lemma 3). On the other hand, the weak opposition is naturally defined in our framework as $\langle A, B\rangle^{\triangledown} = \langle ((M \cup \overline{M}) \setminus B)^{\Downarrow}, ((M \cup \overline{M}) \setminus B)^{\Downarrow\Uparrow}\rangle$. Thus, the role of weak opposition in our approach is trivial since it can be described as follows:

- If $\langle A_1, B_1\rangle$ is an atom in $\mathfrak{B}^{\sharp}(\mathbb{K})$ and there is another atom $\langle A_2, B_2\rangle$ such that $B_1 = \overline{B}_2$ (recall that B_1 and B_2 are full consistent sets in this case), then $\langle A_1, B_1\rangle^{\triangledown} = \langle A_2, B_2\rangle$.

- $\langle \varnothing, M \cup \overline{M}\rangle^{\triangledown} = \langle G, G^{\Uparrow}\rangle$ and $\langle A, B\rangle^{\triangledown} = \langle \varnothing, M \cup \overline{M}\rangle$ otherwise.

As in the general case, in mixed concept lattices, the weak negation satisfies the principle of excluded middle. The principle of contradiction is ensured for the weak negation of $\wedge$-irreducible elements (mixed concept lattices are $\wedge$-complemented) but not necessarily in other elements. See, for instance, the mixed concept lattice depicted in Figure 3, where the weak negation of ℓ_7 is ℓ_9 and $\ell_7 \wedge \ell_9 = \ell_2 \neq \bot$. Analogously, the law of double negation can be ensured for $\wedge$-irreducible elements but not in general, e.g. $\ell_2^{\triangle\triangle} = \top^{\triangle} = \bot$ in the same example.

Obviously, the above discussion concerning weak negation, weak opposition and the classical laws related to negation can also be stated for the concept lattice of a dichotomous context.

Since neither the diamond, nor the pentagon obey both the atomic $\vee$-simplification law and the fact that mixed concept lattices are atomistic, it is natural to ask how these properties relate to distributivity. In this line, it is easy to prove the following proposition.

Proposition 1. *Any finite boolean lattice is a mixed concept lattice.*

However, the contrary result does not hold: there exist mixed concept lattices embedding a pentagon or a diamond as a sublattice. See, for instance, the lattice depicted in Figure 4.

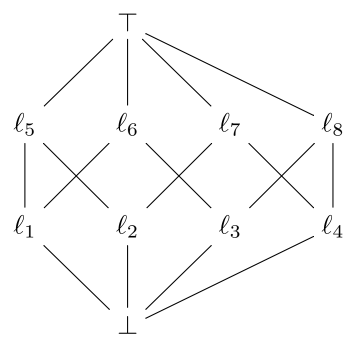

Figure 4: Mixed concept lattice that is neither distributive nor p-algebra.

In [6], Ganter and Kwuida define a negation on the concept level by using *pseudocomplemented lattices*, also known as *p-algebras*. In a bounded lattice, the pseudocomplement of an element ℓ_1, if it exists, is the largest element ℓ_1^* such that $\ell_1 \wedge \ell_2 = \bot$ iff $\ell_2 \leq \ell_1^*$. Thus, a p-algebra is a bounded lattice where any element has a pseudocomplement. In a concept lattice, the pseudocomplement of a concept $\langle A, B \rangle$ is the most general concept (when it exists) that contradicts $\langle A, B \rangle$. Such a pseudocomplement may be interpreted as a negation of the concept. The following proposition can be proved by identifying the pseudocomplement of an element with its opposite one.

Proposition 2. *Any finite atomistic p-algebra is a mixed concept lattice.*

In addition, in a mixed concept lattice, for any $\wedge$-irreducible element, its opposite element is also its pseudocomplement. However, not every mixed concept lattice is a p-algebra. For instance, the lattice depicted in Figure 4 is a mixed concept lattice, but the atoms have no pseudocomplement.

7. Conclusions and future work

In this work, the foundations of formal concept analysis have been extended to consider negative information on the attribute level. Previous work in the literature sticks to the original framework and introduces some alternative strategies to adapt classical methods and techniques to the richer environment. Such an approach hides the semantics of mixed information and may lead to suboptimal algorithmic approaches, avoiding the use of the direct results emerging in the new framework. Contrariwise, here we propose to modify the algebraic foundations to exploit all the advantages of the inherent semantics of the negation.

Such a challenge requires a deep theoretical study covering the definition of some of the main elements of FCA and, even more, the connections among them. In this work we focused on the two pillars of FCA: formal context and concept lattice. One key point in FCA is that, given an arbitrary lattice, there always exists a formal context whose concept lattice is isomorphic to it. In contrast to this situation in standard FCA, when negative attributes are considered, for some lattices such an isomorphism to a mixed concept lattice cannot be built. The main result of this work, presented at the end of the paper, is the theorem which allows to characterize those lattices that are isomorphic to a mixed concept lattice.

As future work, the extended framework and its new results will pave the way toward direct algorithms to compute mixed concept lattices. They will be inspired by the classical ones but take into account the advantages provided by the algebraic results presented here, allowing an increase in performance. As a first step in this direction, in [20], we make a review of the most relevant traditional algorithm for mining concept lattices and propose a preliminary extended versions of all of them to compute mixed concept lattices. Our intention is to study the applicability of the results given in [1] to the problem of mining mixed concept lattices.

On the other side, this is only the first step in the development of a complete study of mixed formal concept analysis. Regarding mixed concept lattices, the consequences of special cases where additional properties are present have to be studied.

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References

[1] L. Beaudou, A. Mary, L. Nourine, Algorithms for k-meet-semidistributive lattices, Theoretical Computer Science (2016) `doi:10.1016/j.tcs.2015.10.029`.

[2] K. Bertet, B. Monjardet, The multiple facets of the canonical direct unit implicational basis, Theoretical Computer Sciences 411 (22-24) (2010) 2155–2166. `doi:10.1016/j.tcs.2009.12.021`.

[3] J. Boulicaut, A. Bykowski, B. Jeudy, Towards the tractable discovery of association rules with negations, Advances in Soft Computing (2001) 425–434. `doi:10.1007/978-3-7908-1834-5_39`.

[4] B. Ganter, Two basic algorithms in concept analysis. Lecture Notes in Computer Science 5986 (2010) 312–340. doi:10.1007/978-3-642-11928-6_22.

[5] B. Ganter, L. Kwuida, Finite distributive concept algebras, Order 23 (2-3) (2006) 235–248. doi:10.1007/s11083-006-9045-x.

[6] B. Ganter, L. Kwuida, Which concept lattices are pseudocomplemented?, Lecture Notes in Computer Science 3403 (2005) 408–416. doi:10.1007/978-3-540-32262-7_29.

[7] B. Ganter, R. Wille, Formal Concept Analysis: Mathematical Foundations, Springer, 1997.

[8] G. Gasmi, S. B. Yahia, E. M. Nguifo, S. Bouker, Extraction of association rules based on literalsets, Lecture Notes in Computer Science 4654 (2007) 293–302. doi:10.1007/978-3-540-74553-2_27.

[9] G. Grätzer, General Lattice Theory, Birkhäuser Verlag, 1998.

[10] J. L. Guigues, V. Duquenne, Familles minimales d'implications informatives résultant d'un tableau de données binaires, Mathématiques et Sciences Humaines 95 (1986) 5–18. URL http://eudml.org/doc/94331

[11] S. O. Kuznetsov, Machine learning and formal concept analysis. Lecture Notes in Computer Science 2961 (2004) 287–312. doi:10.1007/978-3-540-24651-0_25.

[12] S. O. Kuznetsov, A. Revenko, Interactive error correction in implicative theories, International Journal of Approximate Reasoning 63 (2015) 89–100. doi:10.1016/j.ijar.2015.06.004.

[13] L. Kwuida, A. Tepavcevic, B. Seselja, Negation in contextual logic, Lecture Notes in Computer Science 3127 (2004) 227–241. doi:10.1007/978-3-540-27769-9_15.

[14] R. Missaoui, L. Nourine, Y. Renaud, Generating positive and negative exact rules using formal concept analysis: Problems and solutions. of Lecture Notes in Computer Science 4933 (2008) 169–181. doi:10.1007/978-3-540-78137-0_13.

[15] R. Missaoui, L. Nourine, Y. Renaud, An inference system for exhaustive generation of mixed and purely negative implications from purely positive ones. CEUR Workshop Proceedings 672 (2010) 271–282.

[16] R. Missaoui, L. Nourine, Y. Renaud, Computing implications with negation from a formal context, Fundamenta Informaticae 115 (4) (2012) 357–375. doi:10.3233/FI-2012-659.

[17] A. Revenko, S. O. Kuznetsov, Finding errors in new object intents. CEUR Workshop Proceedings 972 (2012) 151–162.

[18] J. M. Rodríguez-Jiménez, P. Cordero, M. Enciso, A. Mora, Negative attributes and implications in formal concept analysis, Procedia Computer Science 31 (2014) 758–765. doi:10.1016/j.procs.2014.05.325.

[19] J. M. Rodríguez-Jiménez, P. Cordero, M. Enciso, A. Mora, A generalized framework to consider positive and negative attributes in formal concept analysis. CEUR Workshop Proceedings 1252 (2014) 267–278.

[20] J. M. Rodriguez-Jimenez, P. Cordero, M. Enciso, A. Mora, Data mining algorithms to compute mixed concepts with negative attributes: an application to breast cancer data analysis, Mathematical Methods in the Applied Sciences (2016) doi:10.1002/mma.3814.

[21] M. Wild, The joy of implications, aka pure horn functions: mainly a survey, Theoretical Computer Science (2016) doi:10.1016/j.tcs.2016.03.018.

[22] R. Wille, Boolean concept logic, Lecture Notes in Computer Science 1867 (2000) 317–331. doi:10.1007/10722280_22.

[23] W. Yao, S.E. Han, R. Wang, Lattice-theoretic contexts and their concept lattices via Galois ideals, Information Sciences 339 (2016) 1–18 doi:10.1016/j.ins.2015.12.028.