TECHNISCHE UNIVERSITÄT DRESDEN

DEDUCTION SYSTEMS

Tableau Procedures II

Sebastian Rudolph

DRESDEN concept
Exzellenz aus Wissenschaft und Kultur

Agenda

- Recap Tableau Calculus
- Tableau with $\mathcal{ALC}$ TBoxes
- Tableau for $\mathcal{ALC}$ Knowledge Bases
- Extension by Inverse Roles
- Extension by Functional Roles
- Model Construction with Unravelling
- Summary

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- check of satisfiability of C by construction of an abstraction of a model $\mathcal{I}$ such that $C^{\mathcal{I}} \neq \emptyset$

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- tableau construction successful if no rules applicable and no contradiction
- C is satisfiable iff there is a successful tableau construction

Tableau Rules for $\mathcal{ALC}$ Concepts

$\sqcap$-rule: For an $v \in V$ with $C \sqcap D \in L(v)$ and $\{C, D\} \not\subseteq L(v)$, let $L(v) := L(v) \cup \{C, D\}$.

$\sqcup$-rule: For an $v \in V$ with $C \sqcup D \in L(v)$ and $\{C, D\} \cap L(v) = \emptyset$, choose $X \in \{C, D\}$ and let $L(v) := L(v) \cup \{X\}$.

$\exists$-rule: For an $v \in V$ with $\exists r.C \in L(v)$ such that there is no r-successor v' of v with $C \in L(v')$, let $V = V \cup \{v'\}$, $E = E \cup \{\langle v, v' \rangle\}$, $L(v') := \{C\}$ and $L(v, v') := \{r\}$ for v' a new node.

$\forall$-rule: For $v, v' \in V$, v' r-neighbor of v, $\forall r.C \in L(v)$ and $C \notin L(v')$, let $L(v') := L(v') \cup \{C\}$.

Tableau Algorithm Example

$$C = \exists r.(A \sqcup \exists r.B) \sqcap \exists r.\neg A \sqcap \forall r.(\neg A \sqcap \forall r.(\neg B \sqcup A))$$

u

$$L(u) = \{C\}$$

Tableau Algorithm Example

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$$L(u) = \{C, \exists r.(A \sqcup \exists r.B), \exists r.\neg A, \forall r.(\neg A \sqcap \forall r.(\neg B \sqcup A))\}$$

Tableau Algorithm Example

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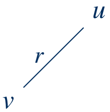

$$L(u) = \{C, \exists r.(A \sqcup \exists r.B), \exists r.\neg A, \forall r.(\neg A \sqcap \forall r.(\neg B \sqcup A))\}$$

$$L(v) = \{A \sqcup \exists r.B\}$$

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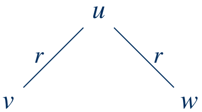

$$\begin{aligned} L(u) &= \{C, \exists r.(A \sqcup \exists r.B), \\ &\quad \exists r.\neg A, \forall r.(\neg A \sqcap \forall r.(\neg B \sqcup A))\} \\ L(v) &= \{A \sqcup \exists r.B\} \\ L(w) &= \{\neg A\} \end{aligned}$$

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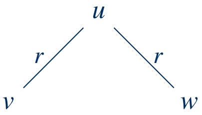

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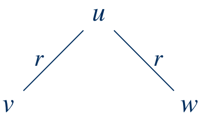

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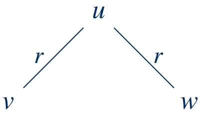

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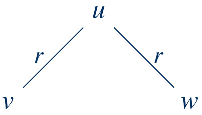

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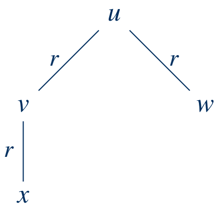

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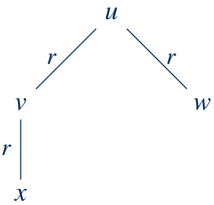

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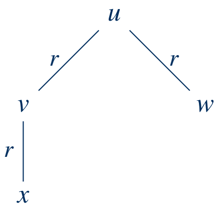

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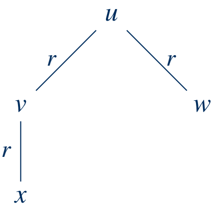

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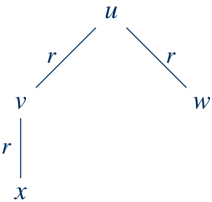

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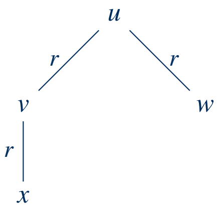

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Tableau Algorithm Example

the model $\mathcal{I}$ constructed by the algorithm is the following:

$$\begin{aligned}
\Delta^{\mathcal{I}} &= \{u, v, w, x\} \\
A^{\mathcal{I}} &= \{x\} \\
B^{\mathcal{I}} &= \{x\} \\
r^{\mathcal{I}} &= \{\langle u, v\rangle, \langle u, w\rangle, \langle v, x\rangle\}
\end{aligned}$$

Check that indeed $C^{\mathcal{I}} = \{u\}$, given the defined semantics of $\mathcal{ALC}$

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Tableau Algorithm for TBoxes

We extend the tableau algorithm to capture $\mathcal{ALC}$ TBoxes

- a TBox contains axioms (GCIs) of the form $C \sqsubseteq D$
- assumption: occurrences of $C \equiv D$ have been replaced by $C \sqsubseteq D$ and $D \sqsubseteq C$
- every GCI is equivalent to $\top \sqsubseteq \neg C \sqcup D$

We can compress the whole TBox into one axiom (we say we "internalize" it):

$$\mathcal{T} = \{C_i \sqsubseteq D_i \mid 1 \leq i \leq n\}$$

is equivalent to:

$$\mathcal{T}' = \{\top \sqsubseteq \bigsqcap_{1 \leq i \leq n} \neg C_i \sqcup D_i\}$$

Let $C_{\mathcal{T}}$ be the concept on the rhs of the GCI in NNF.

Tableau Algorithm for TBoxes

We extend the rules of the $\mathcal{ALC}$ tableau algorithm with the rule:

$\mathcal{T}$ rule: For an arbitrary $v \in V$ with $C_\mathcal{T} \notin L(v)$,
let $L(v) := L(v) \cup \{C_\mathcal{T}\}$.

Example: Let $\mathcal{T} = A \sqsubseteq \exists r.A$. Is A satisfiable given $\mathcal{T}$?

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solution: we will recognize cycles (that is, repeating node labellings)

Tableau Algorithm for TBoxes

Definition (Blocking)

A node $v \in V$ blocks a node $v' \in V$ directly, if:

1. v' is reachable from v,
2. $L(v') \subseteq L(v)$; and
3. there is no directly blocking node v'' such that v' is reachable from v''.

A node $v' \in V$ is blocked if either

1. v' is blocked directly or
2. there is a directly blocked node v, such that v' is reachable from v.

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A node $v' \in V$ is blocked if either

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The application of the $\exists$ rule is restricted to nodes that are not blocked.

Tableau Algorithm with Blocking

Example: Let $\mathcal{T} = A \sqsubseteq \exists r.A$. Is A satisfiable w.r.t. $\mathcal{T}$?

we obtain the following contradiction-free tableau:

$$L(v_0) = \{A, C_{\mathcal{T}}, \exists r.A\}$$
$$L(v_1) = \{A, C_{\mathcal{T}}, \exists r.A\}$$

wherein v_1 is directly blocked by v_0

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again, the algorithm constructs finite trees

- from a contradiction-free tableau, we can construct a model
- if there is no contradiction-free tableau, there is no model

From the Tableau to the Model

again, we can construct a finite model from a contradiction-free tableau:

$$\Delta^{\mathcal{I}} = \{v_0\}$$
$$A^{\mathcal{I}} = \Delta^{\mathcal{I}}$$
$$r^{\mathcal{I}} = \{\langle v_0, v_0\rangle\}$$

- blocked nodes do not represent elements of the model
- when constructing the model, an edge from a node v to a directly blocked node v' will be “translated” into an “edge” from v to the node, that directly blocks v'

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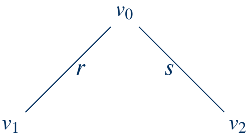
$\rightsquigarrow$ we have the finite model property

$\rightsquigarrow$ constructed model is not necessarily tree-shaped

Tableau Algorithm with Blocking II

Example: Let $\mathcal{T} = A \sqsubseteq \exists r.A \sqcap \exists s.B$. Is A satisfiable w.r.t. $\mathcal{T}$?

We obtain the following contradiction-free tableau:



$$L(v_0) = \{A, C_{\mathcal{T}}, \exists r.A \sqcap \exists s.B, \exists r.A, \exists s.B\}$$
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$$L(v_2) = \{B, C_{\mathcal{T}}, \neg A\}$$

in which v_1 is again directly blocked by v_0

From the Tableau to a Model II

again, we can construct a finite model from a contradiction-free tableau:

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Example: Let $\mathcal{T} = \{A \sqsubseteq B \sqcap \exists r.C, B \equiv C \sqcup D, C \sqsubseteq \exists r.D\}$. Is A satisfiable w.r.t. $\mathcal{T}$?

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Normalization I:
$\mathcal{T}' = \{A \sqsubseteq B, A \sqsubseteq \exists r.C, B \sqsubseteq C \sqcup D, C \sqcup D \sqsubseteq B, C \sqsubseteq \exists r.D\}$

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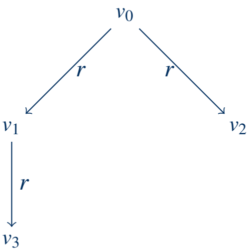
$\mathcal{T}' = \{A \sqsubseteq B, A \sqsubseteq \exists r.C, B \sqsubseteq C \sqcup D, C \sqsubseteq B, D \sqsubseteq B, C \sqsubseteq \exists r.D\}$

$C_{\mathcal{T}} = (\neg A \sqcup B) \sqcap (\neg A \sqcup \exists r.C) \sqcap (\neg B \sqcup C \sqcup D) \sqcap (\neg C \sqcup B) \sqcap (\neg D \sqcup B) \sqcap (\neg C \sqcup \exists r.D)$

Tableau Algorithm Example

$$C_{\mathcal{T}} = (\neg A \sqcup B) \sqcap (\neg A \sqcup \exists r.C) \sqcap (\neg B \sqcup C \sqcup D) \sqcap (\neg C \sqcup B) \sqcap (\neg D \sqcup B) \sqcap (\neg C \sqcup \exists r.D)$$

we obtain the following contradiction-free tableau:



$$L(v_0) = \{A, C_{\mathcal{T}}, \ldots, B, \exists r.C, C, \neg D, \exists r.D\}$$
$$L(v_1) = \{C, C_{\mathcal{T}}, \ldots, \neg A, B, \exists r.D\}$$
$$L(v_2) = \{D, C_{\mathcal{T}}, \ldots, \neg A, \neg C, B\}$$
$$L(v_3) = \{D, C_{\mathcal{T}}, \ldots, \neg A, \neg C, B\}$$

Agenda

- Recap Tableau Calculus
- Tableau with $\mathcal{ALC}$ TBoxes
- Tableau for $\mathcal{ALC}$ Knowledge Bases
- Extension by Inverse Roles
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to take an ABox $\mathcal{A}$ into account, initialize G such that

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the tableau rules can then be applied to this initialized graph

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in order to take into account inverse roles, we have to make the following changes

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   - v' is an r-successor of v or
   - v is an r^--successor of v'
3. replace the term “r-successor” in the ∀- and the ∃-rule with “r-neighbor”

the ∃-rule still generates

- an r-successor for a concept $\exists r.C$ (if no fitting neighbor exists yet)
- an r^--successor for a concept $\exists r^-.C$ (if no fitting neighbor exists yet)

Tableau Example with Inverses

Example: is A satisfiable w.r.t. $\mathcal{T}$?

$$\mathcal{T} = \{A \equiv \exists r^{-}.A \sqcap (\forall r.(\neg A \sqcup \exists s.B))\}$$

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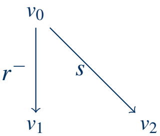

$$L(v_0) = \{A, C_\mathcal{T}, \exists r^-.A, \forall r.(\neg A \sqcup \exists s.B),$$
$$\neg A \sqcup \exists s.B, \exists s.B\}$$
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v_0 blocks v_1

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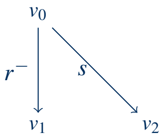

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Is the algorithm thus correct?

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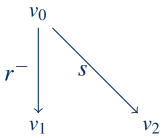

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$$L(v_1) = \{A, C_{\mathcal{T}}, \exists r^{-}.A, \forall r.(\neg A \sqcup \exists s.B)\}$$

$$L(v_2) = \{B, C_{\mathcal{T}}, \neg A, \forall r^{-}.(\neg A)\}$$

v_0 blocks v_1

Is the algorithm thus correct? No!

Tableau Example with Inverses II

Example: Is $C \sqcap \exists s.C$ satisfiable w.r.t. $\mathcal{T}$?

$$\mathcal{T} = \{\top \sqsubseteq \forall r^-.(\forall s^-.(\neg C)) \sqcap \exists r.C\}$$

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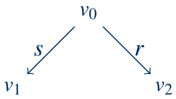

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$$L(v_0) = \{C, \exists s.C, C_\mathcal{T}, \forall r^-.(\forall s^-.(\neg C)), \exists r.C, \forall s^-.(\neg C)\}$$

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v_0 blocks v_1 and v_2

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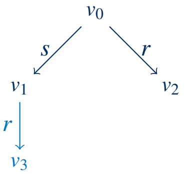
Example: Is $C \sqcap \exists s.C$ satisfiable w.r.t. $\mathcal{T}$?



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v_0 blocks v_1 and v_2 but

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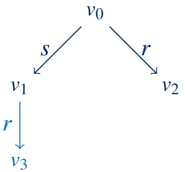

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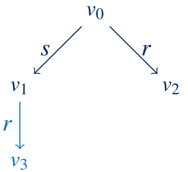

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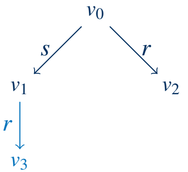

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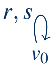
correctness can be retained by replacing subset blocking with equality blocking i.e., replace $L(v') \subseteq L(v)$ by $L(v') = L(v)$ in the blocking condition

Model Construction for Tableau Example with Inverses II

We cannot build a cyclic model as we could up to now !

Example: Is $C \sqcap \exists s.C$ satisfiable w.r.t. $\mathcal{T}$?



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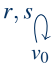

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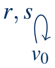

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Example with Inverses & Equality Blocking

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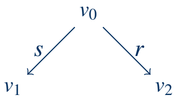

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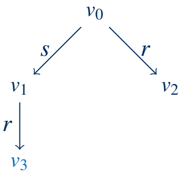

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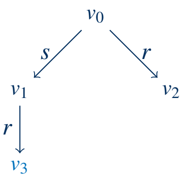
v_1 blocks v_3 but $\forall$-rule applicable

Example with Inverses & Equality Blocking

Example: Is $C \sqcap \exists s.C$ satisfiable w.r.t. $\mathcal{T}$?



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Example with Inverses & Equality Blocking

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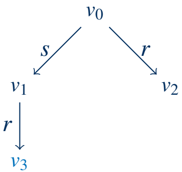

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~~v_1 blocks v_3 but $\forall$-rule applicable~~

Now unsatisfiability is recognized!

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Example: is A satisfiable w.r.t. $\mathcal{T}$?
Note: $\top \sqsubseteq \leqslant 1 f$ is equivalent to Func(f)

$$\mathcal{T} = \{A \sqsubseteq \exists f.B \sqcap \exists f.(\neg B), \top \sqsubseteq \leqslant 1 f\}$$

Tableau with Functional Roles

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Note: $\top \sqsubseteq \leqslant 1f$ is equivalent to Func(f)

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$$C_{\mathcal{T}} = (\neg A \sqcup (\exists f.B \sqcap \exists f.(\neg B))) \sqcap \leqslant 1f$$

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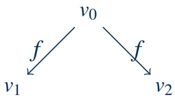


$$\mathcal{T} = \{A \sqsubseteq \exists f.B \sqcap \exists f.(\neg B), \top \sqsubseteq \leqslant 1 f\}$$

$$C_{\mathcal{T}} = (\neg A \sqcup (\exists f.B \sqcap \exists f.(\neg B))) \sqcap \leqslant 1 f$$

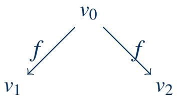
$$L(v_0) = \{A, C_{\mathcal{T}}, \ldots, \exists f.B, \exists f.(\neg B), \leqslant 1 f\}$$

$$L(v_1) = \{B, C_{\mathcal{T}}, \ldots, \neg A, \leqslant 1 f\}$$

$$L(v_2) = \{\neg B, C_{\mathcal{T}}, \ldots, \neg A, \leqslant 1 f\}$$

Tableau with Functional Roles

Example: is A satisfiable w.r.t. $\mathcal{T}$?
Note: $\top \sqsubseteq \leqslant 1 f$ is equivalent to Func(f)



$$\mathcal{T} = \{A \sqsubseteq \exists f.B \sqcap \exists f.(\neg B), \top \sqsubseteq \leqslant 1 f\}$$
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functionality requires $v_1 = v_2$!

$\leadsto$ we need a new tableau rule for treating functional roles

Tableau Rules for $\mathcal{ALCIF}$ Concepts and TBoxes

$\sqcap$-rule:	For an $v \in V$ with $C \sqcap D \in L(v)$ and $\{C, D\} \not\subseteq L(v)$, let $L(v) := L(v) \cup \{C, D\}$.
$\sqcup$-rule:	For an $v \in V$ with $C \sqcup D \in L(v)$ and $\{C, D\} \cap L(v) = \emptyset$, choose $X \in \{C, D\}$ and let $L(v) := L(v) \cup \{X\}$.
$\exists$-rule:	For a non-blocked $v \in V$ with $\exists r.C \in L(v)$ such that there is no r-neighbor v' of v with $C \in L(v')$, let $V = V \cup \{v'\}$, $E = E \cup \{\langle v, v' \rangle\}$, $L(v') := \{C\}$ and $L(v, v') := \{r\}$ for v' a new node.
$\forall$-rule:	For $v, v' \in V$, v' r-neighbor of v, $\forall r.C \in L(v)$ and $C \notin L(v')$, let $L(v') := L(v') \cup \{C\}$.
$\leqslant 1$-rule:	For a functional role f and a $v \in V$ with two f-neighbors v_1 and v_2, execute merge(v_1, v_2).
$\mathcal{T}$-rule:	For a $v \in V$ with $C_\mathcal{T} \notin L(v)$, let $L(v) := L(v) \cup \{C_\mathcal{T}\}$.

Merging Nodes

we define merge(v_1, v_2) as follows:

- if v_1 is an ancestor of v_2,
  let $v_i = v_1$ and $v_o = v_2$;
- otherwise let $v_i = v_2$ and $v_o = v_1$.

let $L(v_i) = L(v_i) \cup L(v_o)$ and execute prune(v_o).

where prune(v_o) is defined as:

- $V_o = \{v \mid v$ belongs to the subtree with root $v_o\}$,
- let $V = V \setminus V_o$ and $E = E \setminus \{\langle v, v_o\rangle \mid v_o \in V_o, \langle v, v_o\rangle \in E\}$.

Tableau with Functional Roles

Example: Is $\exists f.A$ satisfiable w.r.t. $\mathcal{T}$?

$$\mathcal{T} = \{A \sqsubseteq \exists f.A, \top \sqsubseteq \leqslant 1 f^{-}\}$$

Tableau with Functional Roles

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Tableau with Functional Roles

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$$C_{\mathcal{T}} = (\neg A \sqcup \exists f.A) \sqcap \leqslant 1 f^{-}$$

$$L(v_0) = \{\exists f.A, C_{\mathcal{T}}, \neg A, \leqslant 1 f^{-}\}$$

$$L(v_1) = \{A, C_{\mathcal{T}}, \exists f.A, \leqslant 1 f^{-}\}$$

$$L(v_2) = \{A, C_{\mathcal{T}}, \exists f.A, \leqslant 1 f^{-}\}$$

Tableau with Functional Roles

Example: Is $\exists f.A$ satisfiable w.r.t. $\mathcal{T}$?

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v_1 blocks v_2, but cyclic model construction does not work (functionality violated)!

Agenda

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- Summary

Unravelling

goal: we build an infinite model
How? Every blocked node is replaced by a subtree whose root is the corresponding blocking node.

$$v_0 \xrightarrow{f} v_1 \xrightarrow{f} v_2$$

$$L(v_0) = \{\exists f.A, C_{\mathcal{T}}, \neg A, \leqslant 1 f^-\}$$

$$L(v_1) = \{A, C_{\mathcal{T}}, \exists f.A, \leqslant 1 f^-\}$$

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Unravelling

goal: we build an infinite model
How? Every blocked node is replaced by a subtree whose root is the corresponding blocking node.

$$\begin{array}{c} v_0 \\ f \big\downarrow \\ v_1 \\ f \big\downarrow \\ v_1' \\ f \big\downarrow \\ v_1'' \\ f \big\downarrow \end{array}$$

$$L(v_0) = \{\exists f.A, C_{\mathcal{T}}, \neg A, \leqslant 1 f^-\}$$

$$L(v_1) = \{A, C_{\mathcal{T}}, \exists f.A, \leqslant 1 f^-\}$$

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v_1 blocks v_2

Blocking: Inverse and Functional Roles

Example: Is $\neg C \sqcap \exists f^{-}.D$ satisfiable w.r.t. $\mathcal{T}$?

$$\mathcal{T} = \{D \sqsubseteq C \sqcap \exists f.(\neg C) \sqcap \exists f^{-}.D, \top \sqsubseteq \leqslant 1 f\}$$

Blocking: Inverse and Functional Roles

Example: Is $\neg C \sqcap \exists f^{-}.D$ satisfiable w.r.t. $\mathcal{T}$?

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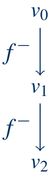

$$L(v_0) = \{\neg C, \exists f^{-}.D, C_{\mathcal{T}}, \ldots, \neg D, \leqslant 1 f\}$$

$$L(v_1) = \{D, C_{\mathcal{T}}, \ldots, C, \exists f.(\neg C), \exists f^{-}.D, \leqslant 1 f\}$$

$$L(v_2) = \{D, C_{\mathcal{T}}, \ldots, C, \exists f.(\neg C), \exists f^{-}.D, \leqslant 1 f\}$$

v_1 blocks v_2(same label)

Blocking: Inverse and Functional Roles

Example: Is $\neg C \sqcap \exists f^-.D$ satisfiable w.r.t. $\mathcal{T}$?

$$\mathcal{T} = \{D \sqsubseteq C \sqcap \exists f.(\neg C) \sqcap \exists f^-.D, \top \sqsubseteq \leqslant 1 f\}$$

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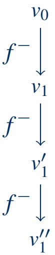

$$L(v_0) = \{\neg C, \exists f^-.D, C_{\mathcal{T}}, \ldots, \neg D, \leqslant 1 f\}$$

$$L(v_1) = \{D, C_{\mathcal{T}}, \ldots, C, \exists f.(\neg C), \exists f^-.D, \leqslant 1 f\}$$

$$L(v_1') = \{D, C_{\mathcal{T}}, \ldots, C, \exists f.(\neg C), \exists f^-.D, \leqslant 1 f\}$$

v_1 blocks v_2(same label) but

$$L(v_1'') = \{D, C_{\mathcal{T}}, \ldots, C, \exists f.(\neg C), \exists f^-.D, \leqslant 1 f\}$$

but we cannot build a model any more (neither cyclic nor infinite)!

Pairwise Blocking

A node x with predecessor x' blocks a node y with predecessor y' directly, if:

1. y is reachable from x,
2. $L(x) = L(y)$, $L(x') = L(y')$ and $L(x', x) = L(y', y)$; and
3. there is no directly blocked node z such that y is reachable from z.

A node $y \in V$ is blocked if either

1. y is directly blocked or
2. there is a directly blocked node x, such that y can be reached from x.

Pairwise Blocking: Inverses and Functional Roles

Example: Is $\neg C \sqcap \exists f^{-}.D$ satisfiable w.r.t. $\mathcal{T}$?

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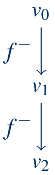

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v_1 cannot block v_2 pairwise

Pairwise Blocking: Inverses and Functional Roles

Example: Is $\neg C \sqcap \exists f^{-}.D$ satisfiable w.r.t. $\mathcal{T}$?

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$$L(v_2) = \{D, C_{\mathcal{T}}, \ldots, C, \exists f.(\neg C), \exists f^{-}.D, \leqslant 1 f\}$$

v_1 cannot block v_2 pairwise

$$L(v_3) = \{\neg C, C_{\mathcal{T}}, \ldots, \neg D, \leqslant 1 f\}$$

Pairwise Blocking: Inverses and Functional Roles

Example: Is $\neg C \sqcap \exists f^{-}.D$ satisfiable w.r.t. $\mathcal{T}$?

$$\mathcal{T} = \{D \sqsubseteq C \sqcap \exists f.(\neg C) \sqcap \exists f^{-}.D, \top \sqsubseteq\leqslant 1 f\}$$

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$$L(v_2) = \{D, C_{\mathcal{T}}, \ldots, C, \exists f.(\neg C), \exists f^{-}.D, \leqslant 1 f\}$$

v_1 cannot block v_2 pairwise

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Pairwise Blocking: Inverses and Functional Roles

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v_1 cannot block v_2 pairwise

$$L(v_3) = \{\neg C, C_{\mathcal{T}}, \ldots, \neg D, \leqslant 1 f\}$$

v_3 is merged into v_1

$$L(v_1) = L(v_1) \cup L(v_3) \supseteq \{\neg D, D\}$$

now the contradiction can be detected

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Summary

- we now have a tableau algorithm for $\mathcal{ALCIF}$ knowledge bases
  - treat the ABox like for $\mathcal{ALC}$
  - number restrictions can be handled similar to functional roles
- termination through cycle detection
  - becomes harder the more expressive the logic gets