Finite and algorithmic model theory (Dresden, Winter 22/23): Exercises 1 (20.10.22 13:00)

1. Try to understand the last proof, skipped during the lecture.
2. Is it true that every satisfiable FO-theory that has an infinite model has also a finite model?
3. Is it true that every satisfiable FO-theory that has a finite model has also an infinite model?
4. Show that if an FO-theory has finite models of arbitrary large sizes[1] then it also has an infinite model.
5. Employ compactness theorem to show that the following property is not-$\mathsf{FO}[\{U\}]$-definable over finite models (with unary relational symbol U): "for a finite $\mathfrak{A}$ both $|U^{\mathfrak{A}}|$ and $|A \setminus U^{\mathfrak{A}}|$ are even".
6. Consider the following finitary analogous of the compactness theorem and show that it is not true.

   False Theorem. *Let $\mathcal{T}$ be an FO theory. If every finite $\mathcal{T}_0 \subseteq \mathcal{T}$ has a finite model then also $\mathcal{T}$ has a finite model.*

7. Here we assume that vocabularies are finite. Show that over finite structures elementary equivalence collapse to isomorphism, i.e. for any two structures $\mathfrak{A}, \mathfrak{B}$ if $\mathfrak{A}$ and $\mathfrak{B}$ satisfy exactly the same first-order formulae then $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic.[2]
8. Does the Löwenheim-Skolem theorem make sense over finite models?

   Definition. *Let $\mathcal{L}$ be a sublogic of FO (i.e. the set of $\mathcal{L}$ sentences is a subset of the set of FO sentences). We say that $\mathcal{L}$ has the* finite model property *(FMP) if every satisfiable formula from $\mathcal{L}$ is also finitely satisfiable (i.e. has a finite model).*

9. (The hardest[3] but most fun!) Prove that if $\mathcal{L} \subseteq \mathsf{FO}$ has the FMP then the satisfiability problem[4] for $\mathcal{L}$ is decidable[5]. Check the hint.

[1] i.e. for each natural number n it has a finite model with more than n elements.
[2] Hint: It suffices to show that $\mathfrak{A}$ can be characterised up-to-isomorphism by an FO formula.
[3] Hint: You must heavily rely on Gödel's completeness theorem and the fact that proofs are finite. Do not care about the running time of your program (actually it might work as long as it wants unless it stops).
[4] Given $\varphi \in \mathcal{L}$ does it have a model?
[5] There is an algorithm solving the problem.

 Compiled on 12/10/2022 at 19:26.