TECHNISCHE UNIVERSITÄT DRESDEN

Computational Logic Group

Foundations of Knowledge Representation

Lecture 4: Description Logics – Syntax and Semantics I

Sebastian Rudolph

based on slides of
Bernardo Cuenca Grau,
Ian Horrocks, and
Przemysław Wałęga

DRESDEN concept
Exzellenz aus Wissenschaft und Kultur

Motivation

Undecidable	FOL sat
Non-Elementary decidable	...
...	...
ExpTime	Datalog sat
PSpace	...
NP	PL sat
P	Horn PL sat

Motivation

Many KR applications do not require full power of FOL

What can we leave out?

- Key reasoning problems should become decidable
- Sufficient expressive power to model application domain

Description Logics are a family of FOL fragments that meet these requirements for many applications:

- Underlying formalisms of modern ontology languages
- Widely-used in bio-medical information systems
- Core component of the Semantic Web

Motivation

Recall our arthritis example:

- A juvenile disease affects only children or teenagers
- Children and teenagers are not adults
- A person is either a child, a teenager, or an adult.
- Juvenile arthritis is a kind of arthritis and a juvenile disease
- Every kind of arthritis damages some joint

The important types of objects given by unary FOL predicates:

juvenile disease, child, teenager, adult, . . .

The types of relationships given by n-ary FOL predicates:

affects, damages (binary predicate), . . .

Motivation

The vocabulary of a Description Logic is composed of

- Unary FOL predicates

  Arthritis, Child, ...

- Binary FOL predicates

  Affects, Damages, ...

- FOL constants

  JohnSmith, MaryJones, JRA, ...

We are already restricting the expressive power of FOL

- No function symbols
- No predicates of arity greater than 2

Motivation

Now, let's take a closer look at the FOL formulas for our example:

$$\forall x.(\mathit{JuvDis}(x) \to \forall y.(\mathit{Affects}(x,y) \to \mathit{Child}(y) \vee \mathit{Teen}(y)))$$
$$\forall x.(\mathit{Child}(x) \vee \mathit{Teen}(x) \to \neg \mathit{Adult}(x))$$
$$\forall x.(\mathit{Person}(x) \to \mathit{Child}(x) \vee \mathit{Teen}(x) \vee \mathit{Adult}(x))$$
$$\forall x.(\mathit{JuvArthritis}(x) \to \mathit{Arthritis}(x) \wedge \mathit{JuvDis}(x))$$
$$\forall x.(\mathit{Arthritis}(x) \to \exists y.(\mathit{Damages}(x,y) \wedge \mathit{Joint}(y))$$

We can find several regularities in these formulas:

- There is an outermost universal quantifier on a single variable x
- They can be split into two parts by the implication symbol

  Each part is a formula with one free variable
- Atomic formulas involving a binary predicate occur only quantified in a syntactically restricted way.

Complexity

Undecidable	FOL-3 sat
|	|
Non-Elementary decidable	...
|	|
...	...
|	|
NExpTime	FOL-2 sat
|	|
ExpTime	Datalog sat
|	|
PSpace	...
|	|
NP	PL sat
|	|
P	Horn PL sat

Motivation

Consider as an example one of our formulas:

$$\forall x.(\mathit{Child}(x) \lor \mathit{Teen}(x) \rightarrow \neg \mathit{Adult}(x))$$

Let's look at all its sub-formulas at each side of the implication

$\mathit{Child}(x)$	Set of all children
$\mathit{Teen}(x)$	Set of all teenagers
$\mathit{Child}(x) \lor \mathit{Teen}(x)$	Set of all objects that are children or teenagers
$\mathit{Adult}(x)$	Set of all adults
$\neg \mathit{Adult}(x)$	Set of all objects that are not adults

Important observations concerning formulas with one free variable:

- Some are atomic (e.g., $\mathit{Child}(x)$)

  do not contain other formulas as subformulas

- Others are complex (e.g., $\mathit{Child}(x) \lor \mathit{Teen}(x)$)

Basic Definitions

Idea: Define operators for constructing complex formulas with one free variable out of simple building blocks

Atomic Concept: Represents an atomic formula with one free variable

$$\mathit{Child} \quad \rightsquigarrow \quad \mathit{Child}(x)$$

Complex concepts (part 1):

- Concept Union ($\sqcup$): applies to two concepts

$$\mathit{Child} \sqcup \mathit{Teen} \quad \rightsquigarrow \quad \mathit{Child}(x) \vee \mathit{Teen}(x)$$

- Concept Intersection ($\sqcap$): applies to two concepts

$$\mathit{Arthritis} \sqcap \mathit{JuvDis} \quad \rightsquigarrow \quad \mathit{Arthritis}(x) \wedge \mathit{JuvDis}(x)$$

- Concept Negation ($\neg$): applies to one concept

$$\neg \mathit{Adult} \quad \rightsquigarrow \quad \neg \mathit{Adult}(x)$$

Motivation

Consider examples with binary predicates:

$$\forall x.(\mathit{Arthritis}(x) \rightarrow \exists y.(\mathit{Damages}(x,y) \land \mathit{Joint}(y))$$
$$\forall x.(\mathit{JuvDis}(x) \rightarrow \forall y.(\mathit{Affects}(x,y) \rightarrow \mathit{Child}(y) \lor \mathit{Teen}(y)))$$

- We have a concept and a binary predicate (called a role) mentioning the concept's free variable
- The role and the concept are connected via conjunction (existential quantification) or implication (universal quantification)
- Nested sub-concepts use a fresh (existentially/universally quantified) variable, and are connected to surrounding concept by exactly one role atom (often called a guard)

Basic Definitions

Atomic Role: Represents an atom with two free variables

$$\mathit{Affects} \quad \rightsquigarrow \quad \mathit{Affects}(x, y)$$

Complex concepts (part 2): apply to an atomic role and a concept

- Existential Restriction:

$$\exists \mathit{Damages}.\mathit{Joint} \quad \rightsquigarrow \quad \exists y.(\mathit{Damages}(x, y) \wedge \mathit{Joint}(y))$$

- Universal Restriction:

$$\forall \mathit{Affects}.(\mathit{Child} \sqcup \mathit{Teen}) \quad \rightsquigarrow \quad \forall y.(\mathit{Affects}(x, y) \rightarrow \mathit{Child}(y) \vee \mathit{Teen}(y))$$

$\mathcal{ALC}$ Concepts

$\mathcal{ALC}$ is the basic description logic

$\mathcal{ALC}$ concepts inductively defined from atomic concepts and roles:

- Every atomic concept is a concept
- $\top$ and $\bot$ are concepts
- If C is a concept, then $\neg C$ is a concept
- If C and D are concepts, then so are $C \sqcap D$ and $C \sqcup D$
- If C a concept and R a role, $\forall R.C$ and $\exists R.C$ are concepts.

Concepts describe sets of objects with certain common features:

$Woman \sqcap \exists hasChild.(\exists hasChild.Person)$	Women with a grandchild
$Disease \sqcap \forall Affects.Child$	Diseases affecting only children
$Person \sqcap \neg\exists owns.DetHouse$	People not owning a detached house
$Man \sqcap \exists hasChild.\top \sqcap \forall hasChild.Man$	Fathers having only sons

Very useful idea for Knowledge Representation !!

General Concept Inclusion Axioms

Recall our example formulas:

$$\forall x.(JuvDis(x) \rightarrow \forall y.(Affects(x, y) \rightarrow Child(y) \vee Teen(y)))$$
$$\forall x.(Child(x) \vee Teen(x) \rightarrow \neg Adult(x))$$
$$\forall x.(Person(x) \rightarrow Child(x) \vee Teen(x) \vee Adult(x))$$
$$\forall x.(JuvArthritis(x) \rightarrow Arthritis(x) \wedge JuvDis(x))$$
$$\forall x.(Arthritis(x) \rightarrow \exists y.(Damages(x, y) \wedge Joint(y))$$

They are of the following form, with $\alpha_C(x)$ and $\alpha_D(x)$ corresponding to $\mathcal{ALC}$ concepts C and D

$$\forall x.(\alpha_C(x) \rightarrow \alpha_D(x))$$

Such sentences are $\mathcal{ALC}$ General Concept Inclusion Axioms (GCIs)

$$C \sqsubseteq D$$

Where C and D are $\mathcal{ALC}$-concepts

General Concept Inclusion Axioms

$$
\begin{aligned}
\forall x.(\mathit{JuvDis}(x) \rightarrow \forall y.(\mathit{Affects}(x,y) \rightarrow \\
\mathit{Child}(y) \vee \mathit{Teen}(y))) \quad & \rightsquigarrow \\
\forall x.(\mathit{Child}(x) \vee \mathit{Teen}(x) \rightarrow \neg \mathit{Adult}(x)) \quad & \rightsquigarrow \\
\forall x.(\mathit{Person}(x) \rightarrow \mathit{Child}(x) \vee \\
\vee \mathit{Teen}(x) \vee \mathit{Adult}(x)) \quad & \rightsquigarrow \\
\forall x.(\mathit{JuvArth}(x) \rightarrow \mathit{Arth}(x) \wedge \mathit{JuvDis}(x)) \quad & \rightsquigarrow \\
\forall x.(\mathit{Arth}(x) \rightarrow \exists y.(\mathit{Damages}(x,y) \wedge \\
\wedge \mathit{Joint}(y)) \quad & \rightsquigarrow
\end{aligned}
$$

Note that we often use $C \equiv D$ as an abbreviation for a symmetrical pair of GCIs $C \sqsubseteq D$ and $D \sqsubseteq C$, e.g.:

$$
\left.\begin{array}{l}
\mathit{Arth} \sqcap \mathit{JuvDis} \sqsubseteq \mathit{JuvArth} \\
\mathit{JuvArth} \sqsubseteq \mathit{Arth} \sqcap \mathit{JuvDis}
\end{array}\right\} \rightsquigarrow \quad \mathit{JuvArth} \equiv \mathit{Arth} \sqcap \mathit{JuvDis}
$$

General Concept Inclusion Axioms

$$
\begin{array}{rcl}
\forall x.(\mathit{JuvDis}(x) \rightarrow \forall y.(\mathit{Affects}(x,y) \rightarrow & & \\
\mathit{Child}(y) \vee \mathit{Teen}(y))) & \leadsto & \mathit{JuvDis} \sqsubseteq \forall \mathit{Affects}.(\mathit{Child} \sqcup \mathit{Teen}) \\
\forall x.(\mathit{Child}(x) \vee \mathit{Teen}(x) \rightarrow \neg \mathit{Adult}(x)) & \leadsto & \\
\forall x.(\mathit{Person}(x) \rightarrow \mathit{Child}(x) \vee & & \\
\vee \mathit{Teen}(x) \vee \mathit{Adult}(x)) & \leadsto & \\
\forall x.(\mathit{JuvArth}(x) \rightarrow \mathit{Arth}(x) \wedge \mathit{JuvDis}(x)) & \leadsto & \\
\forall x.(\mathit{Arth}(x) \rightarrow \exists y.(\mathit{Damages}(x,y) \wedge & & \\
\wedge \mathit{Joint}(y)) & \leadsto &
\end{array}
$$

Note that we often use $C \equiv D$ as an abbreviation for a symmetrical pair of GCIs $C \sqsubseteq D$ and $D \sqsubseteq C$, e.g.:

$$
\left.\begin{array}{l}
\mathit{Arth} \sqcap \mathit{JuvDis} \sqsubseteq \mathit{JuvArth} \\
\mathit{JuvArth} \sqsubseteq \mathit{Arth} \sqcap \mathit{JuvDis}
\end{array}\right\} \leadsto \mathit{JuvArth} \equiv \mathit{Arth} \sqcap \mathit{JuvDis}
$$

General Concept Inclusion Axioms

$$\forall x.(JuvDis(x) \to \forall y.(Affects(x,y) \to Child(y) \vee Teen(y))) \quad \rightsquigarrow \quad JuvDis \sqsubseteq \forall Affects.(Child \sqcup Teen)$$

$$\forall x.(Child(x) \vee Teen(x) \to \neg Adult(x)) \quad \rightsquigarrow \quad Child \sqcup Teen \sqsubseteq \neg Adult$$

$$\forall x.(Person(x) \to Child(x) \vee \vee Teen(x) \vee Adult(x)) \quad \rightsquigarrow$$

$$\forall x.(JuvArth(x) \to Arth(x) \wedge JuvDis(x)) \quad \rightsquigarrow$$

$$\forall x.(Arth(x) \to \exists y.(Damages(x,y) \wedge \wedge Joint(y)) \quad \rightsquigarrow$$

Note that we often use $C \equiv D$ as an abbreviation for a symmetrical pair of GCIs $C \sqsubseteq D$ and $D \sqsubseteq C$, e.g.:

$$\left.\begin{array}{l} Arth \sqcap JuvDis \sqsubseteq JuvArth \\ JuvArth \sqsubseteq Arth \sqcap JuvDis \end{array}\right\} \quad \rightsquigarrow \quad JuvArth \equiv Arth \sqcap JuvDis$$

General Concept Inclusion Axioms

$$\begin{array}{rcl}
\forall x.(\mathit{JuvDis}(x) \rightarrow \forall y.(\mathit{Affects}(x,y) \rightarrow & & \\
\mathit{Child}(y) \vee \mathit{Teen}(y))) & \rightsquigarrow & \mathit{JuvDis} \sqsubseteq \forall \mathit{Affects}.(\mathit{Child} \sqcup \mathit{Teen}) \\
\forall x.(\mathit{Child}(x) \vee \mathit{Teen}(x) \rightarrow \neg \mathit{Adult}(x)) & \rightsquigarrow & \mathit{Child} \sqcup \mathit{Teen} \sqsubseteq \neg \mathit{Adult} \\
\forall x.(\mathit{Person}(x) \rightarrow \mathit{Child}(x) \vee & & \\
\vee \mathit{Teen}(x) \vee \mathit{Adult}(x)) & \rightsquigarrow & \mathit{Person} \sqsubseteq \mathit{Child} \sqcup \mathit{Teen} \sqcup \mathit{Adult} \\
\forall x.(\mathit{JuvArth}(x) \rightarrow \mathit{Arth}(x) \wedge \mathit{JuvDis}(x)) & \rightsquigarrow & \\
\forall x.(\mathit{Arth}(x) \rightarrow \exists y.(\mathit{Damages}(x,y) \wedge & & \\
\wedge \mathit{Joint}(y)) & \rightsquigarrow &
\end{array}$$

Note that we often use $C \equiv D$ as an abbreviation for a symmetrical pair of GCIs $C \sqsubseteq D$ and $D \sqsubseteq C$, e.g.:

$$\left.\begin{array}{l}
\mathit{Arth} \sqcap \mathit{JuvDis} \sqsubseteq \mathit{JuvArth} \\
\mathit{JuvArth} \sqsubseteq \mathit{Arth} \sqcap \mathit{JuvDis}
\end{array}\right\} \rightsquigarrow \mathit{JuvArth} \equiv \mathit{Arth} \sqcap \mathit{JuvDis}$$

General Concept Inclusion Axioms

$$\forall x.(JuvDis(x) \rightarrow \forall y.(Affects(x,y) \rightarrow Child(y) \vee Teen(y))) \quad \rightsquigarrow \quad JuvDis \sqsubseteq \forall Affects.(Child \sqcup Teen)$$

$$\forall x.(Child(x) \vee Teen(x) \rightarrow \neg Adult(x)) \quad \rightsquigarrow \quad Child \sqcup Teen \sqsubseteq \neg Adult$$

$$\forall x.(Person(x) \rightarrow Child(x) \vee \vee Teen(x) \vee Adult(x)) \quad \rightsquigarrow \quad Person \sqsubseteq Child \sqcup Teen \sqcup Adult$$

$$\forall x.(JuvArth(x) \rightarrow Arth(x) \wedge JuvDis(x)) \quad \rightsquigarrow \quad JuvArth \sqsubseteq Arth \sqcap JuvDis$$

$$\forall x.(Arth(x) \rightarrow \exists y.(Damages(x,y) \wedge \wedge Joint(y)) \quad \rightsquigarrow$$

Note that we often use $C \equiv D$ as an abbreviation for a symmetrical pair of GCIs $C \sqsubseteq D$ and $D \sqsubseteq C$, e.g.:

$$\left.\begin{array}{l} Arth \sqcap JuvDis \sqsubseteq JuvArth \\ JuvArth \sqsubseteq Arth \sqcap JuvDis \end{array}\right\} \quad \rightsquigarrow \quad JuvArth \equiv Arth \sqcap JuvDis$$

General Concept Inclusion Axioms

$$
\begin{array}{rcl}
\forall x.(\mathit{JuvDis}(x) \to \forall y.(\mathit{Affects}(x,y) \to & & \\
\mathit{Child}(y) \lor \mathit{Teen}(y))) & \leadsto & \mathit{JuvDis} \sqsubseteq \forall \mathit{Affects}.(\mathit{Child} \sqcup \mathit{Teen}) \\
\forall x.(\mathit{Child}(x) \lor \mathit{Teen}(x) \to \neg \mathit{Adult}(x)) & \leadsto & \mathit{Child} \sqcup \mathit{Teen} \sqsubseteq \neg \mathit{Adult} \\
\forall x.(\mathit{Person}(x) \to \mathit{Child}(x) \lor & & \\
\lor \mathit{Teen}(x) \lor \mathit{Adult}(x)) & \leadsto & \mathit{Person} \sqsubseteq \mathit{Child} \sqcup \mathit{Teen} \sqcup \mathit{Adult} \\
\forall x.(\mathit{JuvArth}(x) \to \mathit{Arth}(x) \land \mathit{JuvDis}(x)) & \leadsto & \mathit{JuvArth} \sqsubseteq \mathit{Arth} \sqcap \mathit{JuvDis} \\
\forall x.(\mathit{Arth}(x) \to \exists y.(\mathit{Damages}(x,y) \land & & \\
\land \mathit{Joint}(y)) & \leadsto & \mathit{Arth} \sqsubseteq \exists \mathit{Damages}.\mathit{Joint}
\end{array}
$$

Note that we often use $C \equiv D$ as an abbreviation for a symmetrical pair of GCIs $C \sqsubseteq D$ and $D \sqsubseteq C$, e.g.:

$$
\left.\begin{array}{l}
\mathit{Arth} \sqcap \mathit{JuvDis} \sqsubseteq \mathit{JuvArth} \\
\mathit{JuvArth} \sqsubseteq \mathit{Arth} \sqcap \mathit{JuvDis}
\end{array}\right\} \leadsto \mathit{JuvArth} \equiv \mathit{Arth} \sqcap \mathit{JuvDis}
$$

Terminological Statements

GCIs allow us to represent a surprising variety of terminological statements

- Sub-type statements

$$\forall x.(\mathit{JuvArth}(x) \rightarrow \mathit{Arth}(x)) \quad \rightsquigarrow \quad \mathit{JuvArth} \sqsubseteq \mathit{Arth}$$

- Full definitions:

$$\forall x.(\mathit{JuvArth}(x) \leftrightarrow \mathit{Arth}(x) \wedge \mathit{JuvDis}(x)) \quad \rightsquigarrow \quad \mathit{JuvArth} \equiv \mathit{Arth} \sqcap \mathit{JuvDis}$$

- Disjointness statements:

$$\forall x.(\mathit{Child}(x) \rightarrow \neg \mathit{Adult}(x)) \quad \rightsquigarrow \quad \mathit{Child} \sqsubseteq \neg \mathit{Adult}$$

- Covering statements:

$$\forall x.(\mathit{Person}(x) \rightarrow \mathit{Adult}(x) \vee \mathit{Child}(x)) \quad \rightsquigarrow \quad \mathit{Person} \sqsubseteq \mathit{Adult} \sqcup \mathit{Child}$$

- Type (domain and range) restrictions:

$$\forall x.(\forall y.(\mathit{Affects}(x,y) \rightarrow \mathit{Arth}(x) \wedge \mathit{Person}(y))) \quad \rightsquigarrow \quad \begin{array}{l} \exists \mathit{Affects}.\top \sqsubseteq \mathit{Arth} \\ \top \sqsubseteq \forall \mathit{Affects}.\mathit{Person} \end{array}$$

Concept Inclusion Axioms & Definitions

Why call $C \sqsubseteq D$ a concept inclusion axiom?

- Intuitively, every object belonging to C should belong also to D
- States that C is more specific than D

Why call it a general concept inclusion axiom?

- It may be interesting to consider restricted forms of inclusion
- E.g., axioms where l.h.s. is atomic are sometimes called definitions
  - A concept definition specifies necessary and sufficient conditions for instances, e.g.:

$$JuvArth \equiv Arth \sqcap JuvDis$$

  - A primitive concept definition specifies only necessary conditions for instances, e.g.:

$$Arth \sqsubseteq \exists Damages.Joint$$

Data Assertions

In description logics, we can also represent data:

$Child(JohnSmith)$	John Smith is a child
$JuvenileArthritis(JRA)$	JRA is a juvenile arthritis
$Affects(JRA, MaryJones)$	Mary Jones is affected by JRA

Usually data assertions correspond to FOL ground atoms.

Often written like this:

$$JohnSmith : Child \qquad (JRA, MaryJones) : Affects$$

In $\mathcal{ALC}$, we have two types of data assertions, for a,b individuals:

$$C(a) \quad \rightsquigarrow \quad \text{C is an } \mathcal{ALC} \text{ concept}$$
$$R(a, b) \quad \rightsquigarrow \quad \text{R is an atomic role}$$

Examples of acceptable data assertions in $\mathcal{ALC}$:

$$\exists hasChild.Teacher(John) \quad \rightsquigarrow \exists y.(hasChild(John, y) \wedge Teacher(y))$$
$$HistorySt \sqcup ClassicsSt(John) \quad \rightsquigarrow HistorySt(John) \vee ClassicsSt(John)$$

DL Knowledge Base: TBox + ABox

An $\mathcal{ALC}$ knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ is composed of:

- A TBox $\mathcal{T}$ (Terminological Component)

  Finite set of GCIs

- An ABox $\mathcal{A}$ (Assertional Component):

  Finite set of assertions

TBox:

$$\mathit{JuvArthritis} \sqsubseteq \mathit{Arthritis} \sqcap \mathit{JuvDisease}$$
$$\mathit{Arthritis} \sqcap \mathit{JuvDisease} \sqsubseteq \mathit{JuvArthritis}$$
$$\mathit{Arthritis} \sqsubseteq \exists \mathit{Damages}.\mathit{Joint}$$
$$\mathit{JuvDisease} \sqsubseteq \forall \mathit{Affects}.(\mathit{Child} \sqcup \mathit{Teen})$$
$$\mathit{Child} \sqcup \mathit{Teen} \sqsubseteq \neg \mathit{Adult}$$

ABox:

$$\mathit{Child}(\mathit{JohnSmith})$$
$$\mathit{JuvArthritis}(\mathit{JRA})$$
$$\mathit{Affects}(\mathit{JRA}, \mathit{MaryJones})$$
$$\mathit{Child} \sqcup \mathit{Teen}(\mathit{MaryJones})$$

Semantics via FOL Translation

$\mathcal{ALC}$ semantics can be defined via translation into FOL:

- Concepts translated as formulas with one free variable

$$
\begin{aligned}
\pi_x(A) &= A(x) & \pi_y(A) &= A(y)\\
\pi_x(\neg C) &= \neg\pi_x(C) & \pi_y(\neg C) &= \neg\pi_y(C)\\
\pi_x(C \sqcap D) &= \pi_x(C) \wedge \pi_x(D) & \pi_y(C \sqcap D) &= \pi_y(C) \wedge \pi_y(D)\\
\pi_x(C \sqcup D) &= \pi_x(C) \vee \pi_x(D) & \pi_y(C \sqcup D) &= \pi_y(C) \vee \pi_y(D)\\
\pi_x(\exists R.C) &= \exists y.(R(x,y) \wedge \pi_y(C)) & \pi_y(\exists R.C) &= \exists x.(R(y,x) \wedge \pi_x(C))\\
\pi_x(\forall R.C) &= \forall y.(R(x,y) \rightarrow \pi_y(C)) & \pi_y(\forall R.C) &= \forall x.(R(y,x) \rightarrow \pi_x(C))
\end{aligned}
$$

- GCIs and assertions translated as sentences

$$
\begin{aligned}
\pi(C \sqsubseteq D) &= \forall x.(\pi_x(C) \rightarrow \pi_x(D))\\
\pi(R(a,b)) &= R(a,b)\\
\pi(C(a)) &= \pi_{x/a}(C)
\end{aligned}
$$

- TBoxes, ABoxes and KBs are translated in the obvious way.

Semantics via FOL Translation

Note redundancy in concept-forming operators:

$$\begin{aligned} \bot &\rightsquigarrow \neg\top \\ C \sqcup D &\rightsquigarrow \neg(\neg C \sqcap \neg D) \\ \forall R.C &\rightsquigarrow \neg(\exists R.\neg C) \end{aligned}$$

These equivalences can be proved using FOL semantics:

$$\begin{aligned} \pi_x(\neg\exists R.\neg C) &= \neg\exists y.(R(x,y) \land \neg\pi_y(C)) \\ &\equiv \forall y.(\neg(R(x,y) \land \neg\pi_y(C))) \\ &\equiv \forall y.(\neg R(x,y) \lor \pi_y(C)) \\ &\equiv \forall y.(R(x,y) \rightarrow \pi_y(C)) \\ &= \pi_x(\forall R.C) \end{aligned}$$

We can define syntax of $\mathcal{ALC}$ using only conjunction and negation operators and the existential role operator.

Direct (Model-Theoretic) Semantics

Direct semantics: An alternative (and convenient) way of specifying semantics

DL interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ is a FOL interpretation over the DL vocabulary:

- Each individual a interpreted as an object $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$.
- Each atomic concept A interpreted as a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$.
- Each atomic role R interpreted as a binary relation $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$.

The mapping $\cdot^{\mathcal{I}}$ is extended to $\top$, $\bot$ and compound concepts as follows:

$$
\begin{aligned}
\top^{\mathcal{I}} &= \Delta^{\mathcal{I}} \\
\bot^{\mathcal{I}} &= \emptyset \\
(\neg C)^{\mathcal{I}} &= \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}} \\
(C \sqcap D)^{\mathcal{I}} &= C^{\mathcal{I}} \cap D^{\mathcal{I}} \\
(C \sqcup D)^{\mathcal{I}} &= C^{\mathcal{I}} \cup D^{\mathcal{I}} \\
(\exists R.C)^{\mathcal{I}} &= \{u \in \Delta^{\mathcal{I}} \mid \exists w \in \Delta^{\mathcal{I}} \text{ s.t. } \langle u, w \rangle \in R^{\mathcal{I}} \text{ and } w \in C^{\mathcal{I}}\} \\
(\forall R.C)^{\mathcal{I}} &= \{u \in \Delta^{\mathcal{I}} \mid \forall w \in \Delta^{\mathcal{I}},\ \langle u, w \rangle \in R^{\mathcal{I}} \text{ implies } w \in C^{\mathcal{I}}\}
\end{aligned}
$$

Direct (Model-Theoretic) Semantics

Consider the interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$

$$
\begin{aligned}
\Delta^{\mathcal{I}} &= \{u, v, w\} \\
\mathit{JuvDis}^{\mathcal{I}} &= \{u\} \\
\mathit{Child}^{\mathcal{I}} &= \{w\} \\
\mathit{Teen}^{\mathcal{I}} &= \emptyset \\
\mathit{Affects}^{\mathcal{I}} &= \{\langle u, w \rangle\}
\end{aligned}
$$

We can then interpret any concept as a subset of $\Delta^{\mathcal{I}}$:

$$
\begin{aligned}
(\mathit{JuvDis} \sqcap \mathit{Child})^{\mathcal{I}} &= \\
(\mathit{Child} \sqcup \mathit{Teen})^{\mathcal{I}} &= \\
(\exists \mathit{Affects}.(\mathit{Child} \sqcup \mathit{Teen}))^{\mathcal{I}} &= \\
(\neg \mathit{Child})^{\mathcal{I}} &= \\
(\forall \mathit{Affects}.\mathit{Teen})^{\mathcal{I}} &=
\end{aligned}
$$

Direct (Model-Theoretic) Semantics

Consider the interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$

$$
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\mathit{Affects}^{\mathcal{I}} &= \{\langle u, w \rangle\}
\end{aligned}
$$

We can then interpret any concept as a subset of $\Delta^{\mathcal{I}}$:

$$
\begin{aligned}
(\mathit{JuvDis} \sqcap \mathit{Child})^{\mathcal{I}} &= \emptyset \\
(\mathit{Child} \sqcup \mathit{Teen})^{\mathcal{I}} &= \\
(\exists \mathit{Affects}.(\mathit{Child} \sqcup \mathit{Teen}))^{\mathcal{I}} &= \\
(\neg \mathit{Child})^{\mathcal{I}} &= \\
(\forall \mathit{Affects}.\mathit{Teen})^{\mathcal{I}} &=
\end{aligned}
$$

Direct (Model-Theoretic) Semantics

Consider the interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$

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\mathit{Child}^{\mathcal{I}} &= \{w\} \\
\mathit{Teen}^{\mathcal{I}} &= \emptyset \\
\mathit{Affects}^{\mathcal{I}} &= \{\langle u, w \rangle\}
\end{aligned}
$$

We can then interpret any concept as a subset of $\Delta^{\mathcal{I}}$:

$$
\begin{aligned}
(\mathit{JuvDis} \sqcap \mathit{Child})^{\mathcal{I}} &= \emptyset \\
(\mathit{Child} \sqcup \mathit{Teen})^{\mathcal{I}} &= \{w\} \\
(\exists \mathit{Affects}.(\mathit{Child} \sqcup \mathit{Teen}))^{\mathcal{I}} &= \\
(\neg \mathit{Child})^{\mathcal{I}} &= \\
(\forall \mathit{Affects}.\mathit{Teen})^{\mathcal{I}} &=
\end{aligned}
$$

Direct (Model-Theoretic) Semantics

Consider the interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$

$$\begin{aligned}
\Delta^{\mathcal{I}} &= \{u, v, w\} \\
\mathit{JuvDis}^{\mathcal{I}} &= \{u\} \\
\mathit{Child}^{\mathcal{I}} &= \{w\} \\
\mathit{Teen}^{\mathcal{I}} &= \emptyset \\
\mathit{Affects}^{\mathcal{I}} &= \{\langle u, w \rangle\}
\end{aligned}$$

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(\neg \mathit{Child})^{\mathcal{I}} &= \\
(\forall \mathit{Affects}.\mathit{Teen})^{\mathcal{I}} &=
\end{aligned}$$

Direct (Model-Theoretic) Semantics

Consider the interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$

$$
\begin{aligned}
\Delta^{\mathcal{I}} &= \{u, v, w\} \\
\mathit{JuvDis}^{\mathcal{I}} &= \{u\} \\
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\mathit{Affects}^{\mathcal{I}} &= \{\langle u, w \rangle\}
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(\neg \mathit{Child})^{\mathcal{I}} &= \{u, v\} \\
(\forall \mathit{Affects}.\mathit{Teen})^{\mathcal{I}} &=
\end{aligned}
$$

Direct (Model-Theoretic) Semantics

Consider the interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$

$$\begin{aligned}
\Delta^{\mathcal{I}} &= \{u, v, w\} \\
\mathit{JuvDis}^{\mathcal{I}} &= \{u\} \\
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\end{aligned}$$

We can then interpret any concept as a subset of $\Delta^{\mathcal{I}}$:

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(\mathit{JuvDis} \sqcap \mathit{Child})^{\mathcal{I}} &= \emptyset \\
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(\exists \mathit{Affects}.(\mathit{Child} \sqcup \mathit{Teen}))^{\mathcal{I}} &= \{u\} \\
(\neg \mathit{Child})^{\mathcal{I}} &= \{u, v\} \\
(\forall \mathit{Affects}.\mathit{Teen})^{\mathcal{I}} &= \{v, w\}
\end{aligned}$$

Direct (Model-Theoretic) Semantics

We can now determine whether $\mathcal{I}$ is a model of ...

- A General Concept Inclusion Axiom $C \sqsubseteq D$:

$$\mathcal{I} \models (C \sqsubseteq D) \quad \text{if} \quad C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$$

- An assertion $C(a)$:

$$\mathcal{I} \models C(a) \quad \text{if} \quad a^{\mathcal{I}} \in C^{\mathcal{I}}$$

- An assertion $R(a, b)$:

$$\mathcal{I} \models R(a, b) \quad \text{if} \quad \langle a^{\mathcal{I}}, b^{\mathcal{I}} \rangle \in R^{\mathcal{I}}$$

- A TBox $\mathcal{T}$, ABox $\mathcal{A}$, and knowledge base:

$$\begin{aligned} \mathcal{I} \models \mathcal{T} &\quad \text{if} \quad \mathcal{I} \models \alpha \text{ for each } \alpha \in \mathcal{T} \\ \mathcal{I} \models \mathcal{A} &\quad \text{if} \quad \mathcal{I} \models \alpha \text{ for each } \alpha \in \mathcal{A} \\ \mathcal{I} \models \mathcal{K} &\quad \text{if} \quad \mathcal{I} \models \mathcal{T} \text{ and } \mathcal{I} \models \mathcal{A} \end{aligned}$$

Direct (Model-Theoretic) Semantics

Consider our previous example interpretation:

$$\Delta^{\mathcal{I}} = \{u, v, w\} \quad \mathit{Affects}^{\mathcal{I}} = \{\langle u, w\rangle\}$$
$$\mathit{JuvDis}^{\mathcal{I}} = \{u\} \qquad \mathit{Child}^{\mathcal{I}} = \{w\} \qquad \mathit{Teen}^{\mathcal{I}} = \emptyset$$

$\mathcal{I}$ is a model of the following axioms:

$$\mathit{JuvDis} \sqsubseteq \exists \mathit{Affects}.\mathit{Child} \quad \rightsquigarrow$$
$$\mathit{Child} \sqsubseteq \neg \mathit{Teen} \quad \rightsquigarrow$$
$$\mathit{JuvDis} \sqsubseteq \forall \mathit{Affects}.\mathit{Child} \quad \rightsquigarrow$$

However $\mathcal{I}$ is not a model of the following axioms:

$$\mathit{JuvDis} \sqsubseteq \exists \mathit{Affects}.(\mathit{Child} \sqcap \mathit{Teen}) \quad \rightsquigarrow$$
$$\neg \mathit{Teen} \sqsubseteq \mathit{Child} \quad \rightsquigarrow$$
$$\exists \mathit{Affects}.\top \sqsubseteq \mathit{Teen} \quad \rightsquigarrow$$

Direct (Model-Theoretic) Semantics

Consider our previous example interpretation:

$$\Delta^{\mathcal{I}} = \{u, v, w\} \quad Affects^{\mathcal{I}} = \{\langle u, w\rangle\}$$
$$JuvDis^{\mathcal{I}} = \{u\} \qquad Child^{\mathcal{I}} = \{w\} \qquad Teen^{\mathcal{I}} = \emptyset$$

$\mathcal{I}$ is a model of the following axioms:

$$JuvDis \sqsubseteq \exists Affects.Child \quad \rightsquigarrow \quad \{u\} \subseteq \{u\}$$
$$Child \sqsubseteq \neg Teen \quad \rightsquigarrow$$
$$JuvDis \sqsubseteq \forall Affects.Child \quad \rightsquigarrow$$

However $\mathcal{I}$ is not a model of the following axioms:

$$JuvDis \sqsubseteq \exists Affects.(Child \sqcap Teen) \quad \rightsquigarrow$$
$$\neg Teen \sqsubseteq Child \quad \rightsquigarrow$$
$$\exists Affects.\top \sqsubseteq Teen \quad \rightsquigarrow$$

Direct (Model-Theoretic) Semantics

Consider our previous example interpretation:

$$\Delta^{\mathcal{I}} = \{u, v, w\} \quad \mathit{Affects}^{\mathcal{I}} = \{\langle u, w\rangle\}$$
$$\mathit{JuvDis}^{\mathcal{I}} = \{u\} \qquad \mathit{Child}^{\mathcal{I}} = \{w\} \qquad \mathit{Teen}^{\mathcal{I}} = \emptyset$$

$\mathcal{I}$ is a model of the following axioms:

$$\mathit{JuvDis} \sqsubseteq \exists \mathit{Affects}.\mathit{Child} \quad \rightsquigarrow \quad \{u\} \subseteq \{u\}$$
$$\mathit{Child} \sqsubseteq \neg \mathit{Teen} \quad \rightsquigarrow \quad \{w\} \subseteq \{u, v, w\}$$
$$\mathit{JuvDis} \sqsubseteq \forall \mathit{Affects}.\mathit{Child} \quad \rightsquigarrow$$

However $\mathcal{I}$ is not a model of the following axioms:

$$\mathit{JuvDis} \sqsubseteq \exists \mathit{Affects}.(\mathit{Child} \sqcap \mathit{Teen}) \quad \rightsquigarrow$$
$$\neg \mathit{Teen} \sqsubseteq \mathit{Child} \quad \rightsquigarrow$$
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Direct (Model-Theoretic) Semantics

Consider our previous example interpretation:

$$\Delta^{\mathcal{I}} = \{u, v, w\} \quad Affects^{\mathcal{I}} = \{\langle u, w\rangle\}$$
$$JuvDis^{\mathcal{I}} = \{u\} \qquad Child^{\mathcal{I}} = \{w\} \qquad Teen^{\mathcal{I}} = \emptyset$$

$\mathcal{I}$ is a model of the following axioms:

$$\begin{aligned} JuvDis \sqsubseteq \exists Affects.Child &\quad\rightsquigarrow\quad \{u\} \subseteq \{u\} \\ Child \sqsubseteq \neg Teen &\quad\rightsquigarrow\quad \{w\} \subseteq \{u, v, w\} \\ JuvDis \sqsubseteq \forall Affects.Child &\quad\rightsquigarrow\quad \{u\} \subseteq \{u, v, w\} \end{aligned}$$

However $\mathcal{I}$ is not a model of the following axioms:

$$\begin{aligned} JuvDis \sqsubseteq \exists Affects.(Child \sqcap Teen) &\quad\rightsquigarrow \\ \neg Teen \sqsubseteq Child &\quad\rightsquigarrow \\ \exists Affects.\top \sqsubseteq Teen &\quad\rightsquigarrow \end{aligned}$$

Direct (Model-Theoretic) Semantics

Consider our previous example interpretation:

$$\Delta^{\mathcal{I}} = \{u, v, w\} \quad \mathit{Affects}^{\mathcal{I}} = \{\langle u, w\rangle\}$$
$$\mathit{JuvDis}^{\mathcal{I}} = \{u\} \quad \mathit{Child}^{\mathcal{I}} = \{w\} \quad \mathit{Teen}^{\mathcal{I}} = \emptyset$$

$\mathcal{I}$ is a model of the following axioms:

$$\mathit{JuvDis} \sqsubseteq \exists \mathit{Affects}.\mathit{Child} \quad \rightsquigarrow \quad \{u\} \subseteq \{u\}$$
$$\mathit{Child} \sqsubseteq \neg \mathit{Teen} \quad \rightsquigarrow \quad \{w\} \subseteq \{u, v, w\}$$
$$\mathit{JuvDis} \sqsubseteq \forall \mathit{Affects}.\mathit{Child} \quad \rightsquigarrow \quad \{u\} \subseteq \{u, v, w\}$$

However $\mathcal{I}$ is not a model of the following axioms:

$$\mathit{JuvDis} \sqsubseteq \exists \mathit{Affects}.(\mathit{Child} \sqcap \mathit{Teen}) \quad \rightsquigarrow \quad \{u\} \not\subseteq \emptyset$$
$$\neg \mathit{Teen} \sqsubseteq \mathit{Child} \quad \rightsquigarrow$$
$$\exists \mathit{Affects}.\top \sqsubseteq \mathit{Teen} \quad \rightsquigarrow$$

Direct (Model-Theoretic) Semantics

Consider our previous example interpretation:

$$\Delta^{\mathcal{I}} = \{u, v, w\} \quad \mathit{Affects}^{\mathcal{I}} = \{\langle u, w\rangle\}$$
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$\mathcal{I}$ is a model of the following axioms:

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$$\mathit{JuvDis} \sqsubseteq \exists \mathit{Affects}.(\mathit{Child} \sqcap \mathit{Teen}) \quad \rightsquigarrow \quad \{u\} \not\subseteq \emptyset$$
$$\neg \mathit{Teen} \sqsubseteq \mathit{Child} \quad \rightsquigarrow \quad \{u, v, w\} \not\subseteq \{w\}$$
$$\exists \mathit{Affects}.\top \sqsubseteq \mathit{Teen} \quad \rightsquigarrow$$

Direct (Model-Theoretic) Semantics

Consider our previous example interpretation:

$$\Delta^{\mathcal{I}} = \{u, v, w\} \quad \mathit{Affects}^{\mathcal{I}} = \{\langle u, w\rangle\}$$
$$\mathit{JuvDis}^{\mathcal{I}} = \{u\} \qquad \mathit{Child}^{\mathcal{I}} = \{w\} \qquad \mathit{Teen}^{\mathcal{I}} = \emptyset$$

$\mathcal{I}$ is a model of the following axioms:

$$\begin{aligned} \mathit{JuvDis} \sqsubseteq \exists \mathit{Affects}.\mathit{Child} &\quad\leadsto\quad \{u\} \subseteq \{u\} \\ \mathit{Child} \sqsubseteq \neg \mathit{Teen} &\quad\leadsto\quad \{w\} \subseteq \{u, v, w\} \\ \mathit{JuvDis} \sqsubseteq \forall \mathit{Affects}.\mathit{Child} &\quad\leadsto\quad \{u\} \subseteq \{u, v, w\} \end{aligned}$$

However $\mathcal{I}$ is not a model of the following axioms:

$$\begin{aligned} \mathit{JuvDis} \sqsubseteq \exists \mathit{Affects}.(\mathit{Child} \sqcap \mathit{Teen}) &\quad\leadsto\quad \{u\} \not\subseteq \emptyset \\ \neg \mathit{Teen} \sqsubseteq \mathit{Child} &\quad\leadsto\quad \{u, v, w\} \not\subseteq \{w\} \\ \exists \mathit{Affects}.\top \sqsubseteq \mathit{Teen} &\quad\leadsto\quad \{u\} \not\subseteq \emptyset \end{aligned}$$