Technische Universität Dresden

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Introduction to Formal Concept Analysis

Exercise Sheet 1, Winter Semester 2017/18

1 Set Theory

Exercise 1 (a piece of recapitulation)

Given the following hints and the universe $M := \{1, 2, 3, 4, 5, 6, 7, 8\}$, compute the sets A, B, C:

(a) $A \cup B = \{2, 3, 4, 5, 6, 7, 8\}$

(b) $B \cup C = \{1, 2, 4, 6, 8\}$

(c) $A \cup C = \{1, 2, 3, 4, 5, 7, 8\}$

(d) $A \cap B = \{2\}$

(e) $B \cap C = \{2, 4, 8\}$

(f) $A \cap C = \{2\}$

Solution:

$$A = \{2, 3, 5, 7\}$$
$$B = \{2, 4, 6, 8\}$$
$$C = \{1, 2, 4, 8\}$$

2 Logic

Exercise 2 (repetition first-order logic)

Formalize the following statements for natural numbers a, b, c, using only multiplication ("$\cdot$"), equality ("$=$") and natural numbers ("0","1","2",...) besides the usual logical symbols ("$\neg$", "$\wedge$", "$\vee$", "$\rightarrow$", "$\leftrightarrow$", "$\forall$", "$\exists$", variables and parentheses):

(i) a divides b.

(ii) a is odd.

(iii) a is common divisor of b and c

(iv) a is the gcd of b and c.

(v) a is a square number.

(vi) a is a prime number.

Solution:

(i) a divides b.
$\exists v \in \mathbb{N} \colon b = a \cdot v$

(ii) a is odd.
$\neg \exists v \in \mathbb{N} \colon a = 2 \cdot v$

(iii) a is common divisor of b and c
$\exists v, w \in \mathbb{N} \colon b = a \cdot v \wedge c = a \cdot w$

(iv) a is the gcd of b and c.
$(\exists v, w \in \mathbb{N} \colon b = a \cdot v \wedge c = a \cdot w) \wedge (\forall u : (\exists v, w \in \mathbb{N} \colon b = u \cdot v \wedge c = u \cdot w) \to (\exists t : a = t \cdot u))$

(v) a is a square number.
$\exists v \in \mathbb{N} \colon v \cdot v = a$

(vi) a is a prime number.
$\neg(a = 1) \wedge \forall v, w \in \mathbb{N} \colon v \cdot w = a \to v = 1 \vee v = a$

3 Derivation Operators and Formal Concepts

Exercise 3 (line diagram)

a) Recall: how is the derivation operator $(\cdot)'$ defined?

b) Let $\mathbb{K} = (G, M, I)$ be a formal context and let $A, B \subseteq G$. Prove the following statements:

1. $A \subseteq B$ implies $B' \subseteq A'$
2. $A \subseteq A''$
3. $A' = A'''$
4. For $C \in G$ and $D \in M$ holds: (C, D) is a formal concept if and only if there is some $E \subseteq G$ such that $C = E''$ and $D = E'$.

<u>**Solution:**</u>

a) For $A \subseteq G$, we defined $A' := \{m \in M \mid \forall g \in A : (g, m) \in I\}$.
For $B \subseteq M$, we defined $B' := \{g \in G \mid \forall m \in B : (g, m) \in I\}$.

b) 1. First note that for any $X \in G$ we obtain $X' = \bigcap_{g \in X} \{m \mid gIm\}$. Then, assuming $A \subseteq B$, we obtain

$$\begin{aligned} B' &= \textstyle\bigcap_{g \in B} \{m \mid gIm\} \\ &= \textstyle\bigcap_{g \in A} \{m \mid gIm\} \cap \bigcap_{g \in B \setminus A} \{m \mid gIm\} \\ &= A' \cap \textstyle\bigcap_{g \in B \setminus A} \{m \mid gIm\} \\ &\subseteq A' \end{aligned}$$

2. We show that for every $g \in G$ with $g \in A$ holds $g \in A''$. From $g \in A$ follows $(g, m) \in I$ for all $m \in A'$. From the latter follows $g \in A''$.

3. We show both $A' \subseteq A'''$ and $A''' \subseteq A'$ using the previous two statements. $A' \subseteq (A')'' = A'''$ follows from the second statement. On the other hand, knowing that $A \subseteq A''$ from the second statement, we can apply the first statement to conclude $A''' = (A'')' \subseteq A'$.

4. We have to show two directions: "if" and "only if". For the "only if" direction, we have to find an appropriate E for a given (C, D). But we can just pick $E = C$, since then we get $C = D' = (C')' = C'' = E''$ as well as $D = C' = E'$. For the "if" direction let E be an arbitrary set and let $C = E''$ and $D = E'$. We now check the conditions $C' = D$ and $C = D'$: we obtain $C' = E''' = E' = D$, on the other hand we obtain $D' = E'' = C$. Hence (C, D) defined this way is a formal concept.

4 Formal Concept Analysis

Exercise 4 (Formal Context)

Regard the following formal context $\mathbb{K}$, given as a cross table:

	needs water to live	lives in water	lives on land	needs chlorophyll to produce food	two seed leaves	one seed leaf	can move around	has limbs	suckles its offspring
Leech	×	×					×		
Bream	×	×					×	×	
Frog	×	×	×				×	×	×
Spike-Weed	×	×		×		×			
Reed	×	×	×	×		×			
Bean	×		×	×	×				
Maize	×		×	×		×			

a) Specify the following sets:

(*i*) {Bean}′

(*ii*) {lives on land}′

(*iii*) {two seed leaves}″

(*iv*) {Frog, Maize}′

(*v*) {needs chlorophyll to produce food, can move around}′

(*vi*) {lives in water, lives on land}′

(*vii*) {needs chlorophyll to produce food, can move around}″

b) Extend $\mathbb{K}$ with both an object and an attribute.

Solution:

(*i*) $\{\text{Bean}\}' = \{\text{needs water to live}, \text{lives on land}, \text{needs chlorophyll to produce food}, \text{two seed leaves}\}$

(*ii*) $\{\text{lives on land}\}' = \{\text{Frog}, \text{Reed}, \text{Bean}, \text{Maize}\}$

(*iii*)
$$\begin{aligned}\{\text{two seed leaves}\}'' &= \{\text{Bean}\}' \\ &= \{\text{needs water to live}, \text{lives on land}, \text{needs chlorophyll to produce food}, \\ &\qquad \text{two seed leaves}\}\end{aligned}$$

(*iv*) $\{\text{Frog, Maize}\}' = \{\text{needs water to live}, \text{lives on land}\}$

(*v*) $\{\text{needs chlorophyll to produce food, can move around}\}' = \emptyset$

(*vi*) $\{\text{lives in water, lives on land}\}' = \{\text{Frog}, \text{Reed}\}$

(*vii*) $\{\text{needs chlorophyll to produce food, can move around}\}'' = \emptyset' = M$