Derivation-Graph-Based Characterizations of Decidable Existential Rule Sets

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Existential Rules: Preliminaries

Instance: set $\mathcal{I} = \{p_1(\vec{t}_1), \ldots, p_n(\vec{t}_n), \ldots\}$ of atoms

Database: finite set $\mathcal{D} = \{p_1(\vec{a}_1), \ldots, p_n(\vec{a}_n)\}$ of ground atoms

Existential Rule: $\forall \vec{x}\vec{y}\beta(\vec{x}, \vec{y}) \rightarrow \exists \vec{z}\alpha(\vec{y}, \vec{z})$ $\vec{y}$ is called "frontier"

Example: $\rho = \mathit{male}(x) \wedge \mathit{human}(x) \rightarrow \exists z(\mathit{female}(z) \wedge \mathit{parent}(z, x))$

Knowledge Base: $\mathcal{K} = (\mathcal{D}, \mathcal{R})$ with $\mathcal{R}$ finite set of existential rules

Derivation: $\delta = \mathcal{D}, (\rho_0, h_0, \mathcal{I}_1), \ldots, (\rho_{n-1}, h_{n-1}, \mathcal{I}_n)$ (forward chaining)

Notation: $\mathcal{D} \xrightarrow{\delta} \mathcal{I}_n$

Boolean Conjunctive Query (BCQ): $\mathcal{Q} = \exists \vec{x}(p_1(\vec{t}_1) \wedge \cdots \wedge p_n(\vec{t}_n))$

Example: $\mathcal{Q} = \exists z(\mathit{female}(z) \wedge \mathit{parent}(z, \mathit{Joe}))$

BCQ Entailment Problem: Does $(\mathcal{D}, \mathcal{R}) \models \mathcal{Q}$ hold?

This holds iff some $\mathcal{I}$ with $\mathcal{D} \xrightarrow{\delta} \mathcal{I}$ is a model of $\mathcal{Q}$

Goals of This Paper

Issue: BCQ Entailment is undecidable in general!

Solution: Restrict $\mathcal{R} \rightsquigarrow$ Decidability

Our Goals:

- Find decidability criteria via *proof-theoretic* analysis
- Using and adapt the existing tool of *derivation graphs*
- Relate established notions to existing concept of *greediness*

Treewidth and Decidability

Bounded Treewidth Sets: A rule set $\mathcal{R}$ is (semantic) **bts** iff $(\mathcal{D}, \mathcal{R})$ has a $\underbrace{\text{universal model of finite treewidth}}_{\text{not explained, irrelevant for this talk.}}$ for any database $\mathcal{D}$.

Theorem (Baget et al. 2011, Feller et al. 2023)

If $\mathcal{R}$ is **bts**, *then BCQ entailment is decidable.*

Note: All rule sets we consider are **bts** $\rightsquigarrow$ BCQ entailment is decidable

Problem: Hard to establish whether given rule set is **bts** (undecidable in general) – search for tools and sufficient criteria establishing "easier" subclasses.

Derivation Graphs

Derivation Graph (Baget et al. 2011): A DAG that keeps track of how facts are derived and newly introduced terms are propagated.

Example: $\mathcal{D} = \{p(a, b)\}$ and

$\mathcal{R} = \{$

1. $p(u, v) \to \exists x, y.p(x, y)$,
2. $p(x, y) \to \exists z.q(y, z)$,
3. $p(x, y) \wedge q(y, z) \to p(y, y)$,
4. $p(y, y) \to q(y, y)$,
5. $q(x, x) \wedge p(y, z) \to r(x, y, z)$.

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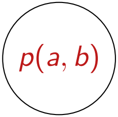

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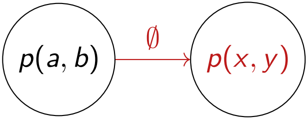

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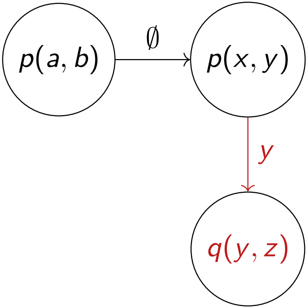

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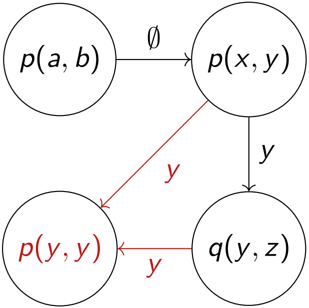

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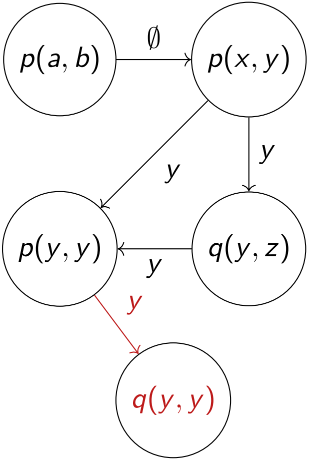

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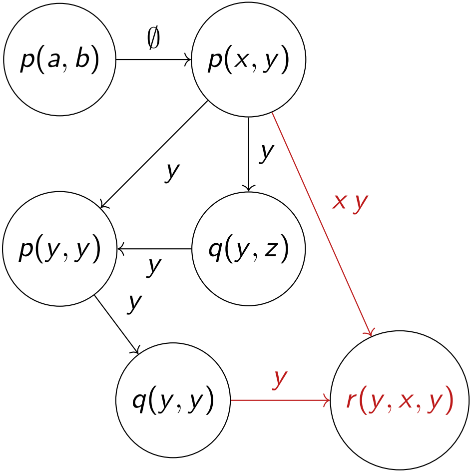
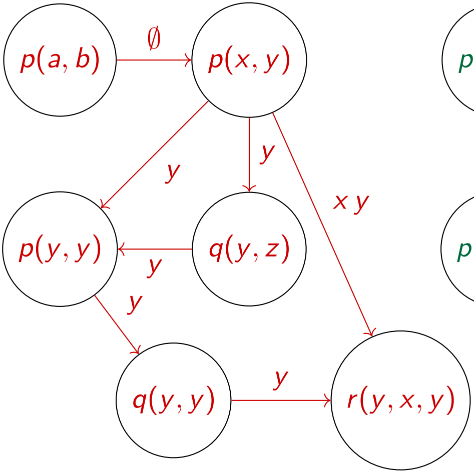
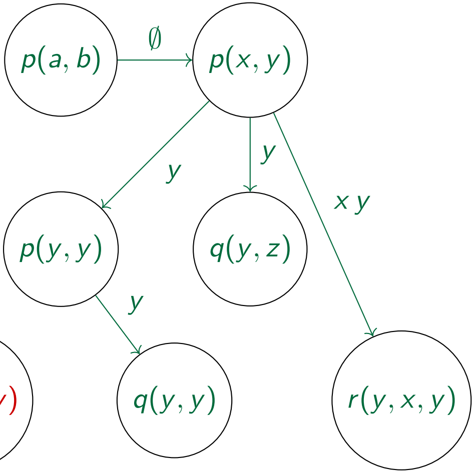


Derivation Graphs and Tree Decompositions

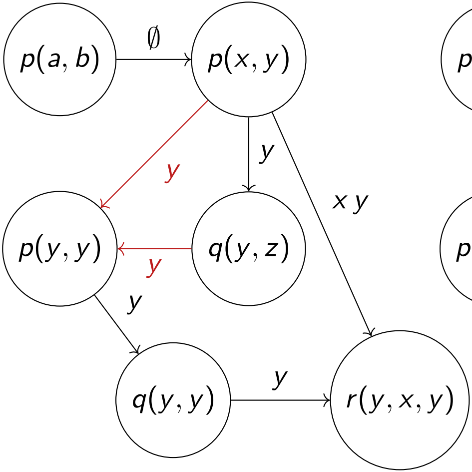
Idea (Baget et al. 2011): If any derivation graph for a rule set $\mathcal{R}$ can be reduced to a tree, then $\mathcal{R}$ is **bts**.

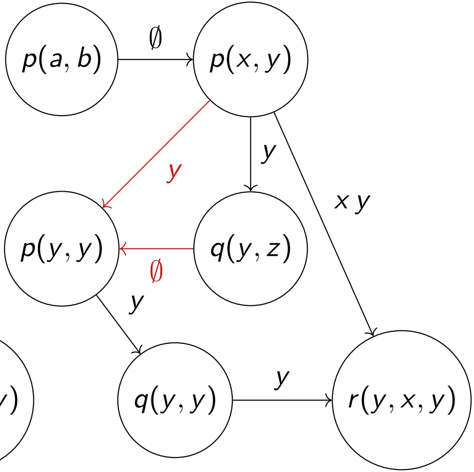




Reduction Operations

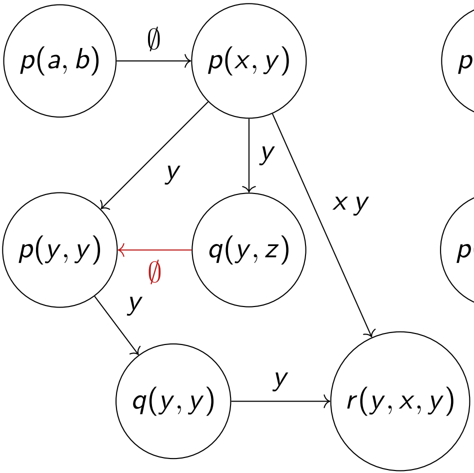
Term Removal: If a term labels two converging arcs, it may be removed from one.

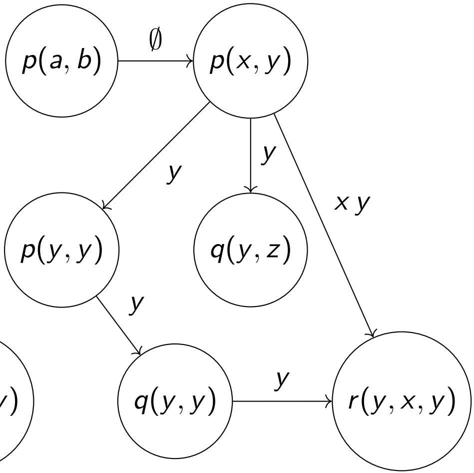




Reduction Operations

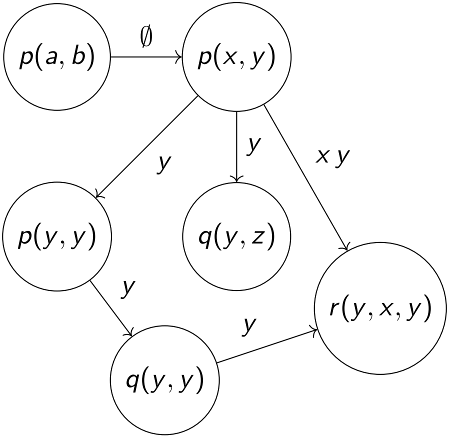
Arc Removal: If an arc is labelled with $\emptyset$, then it may be deleted.

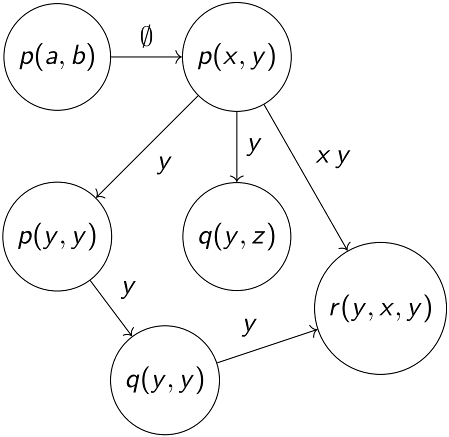




Reduction Operations **NEW!!!**

Cycle Removal: If two arcs (X, Z), (Y, Z) have a "common ancestor" W with $L(X, Z) \cup L(Y, Z) \subseteq terms(W)$, then (1) delete (X, Z), (Y, Z), (2) add (W, Z) with $L(W, Z) = Lab(X, Z) \cup L(Y, Z)$.





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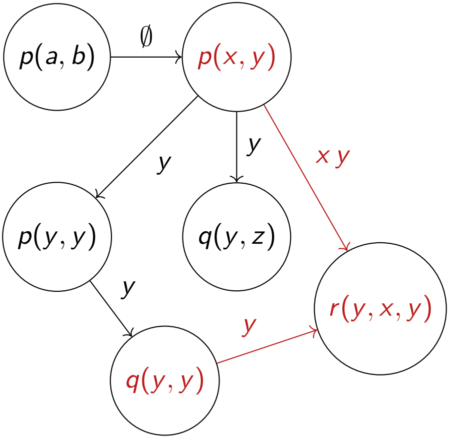

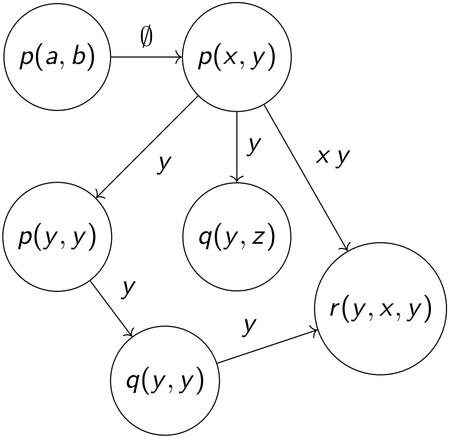

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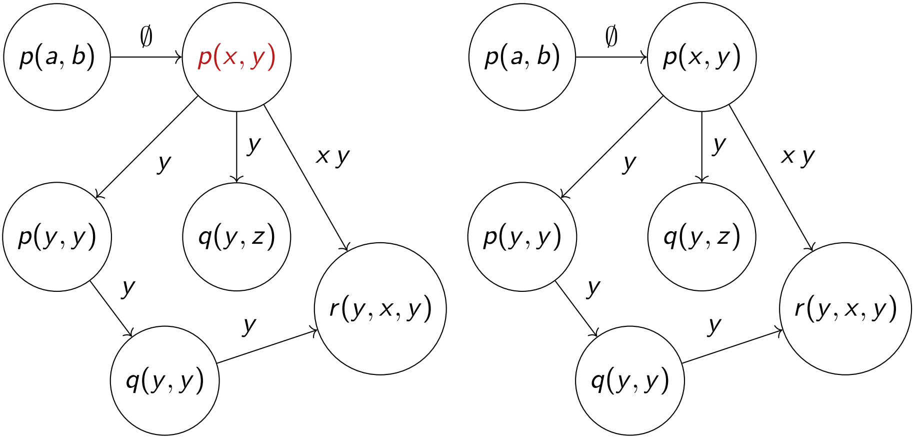

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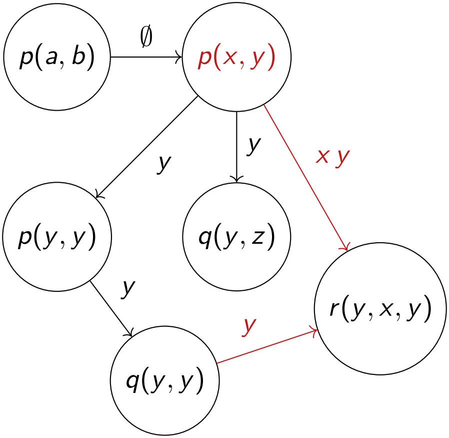

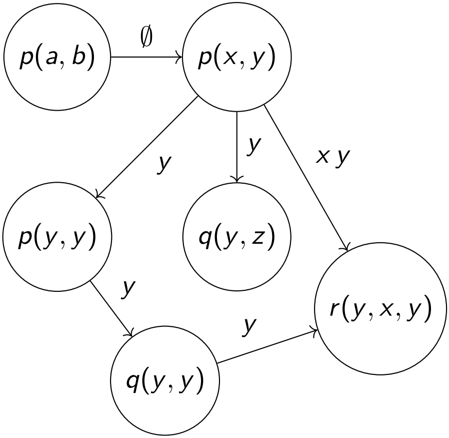

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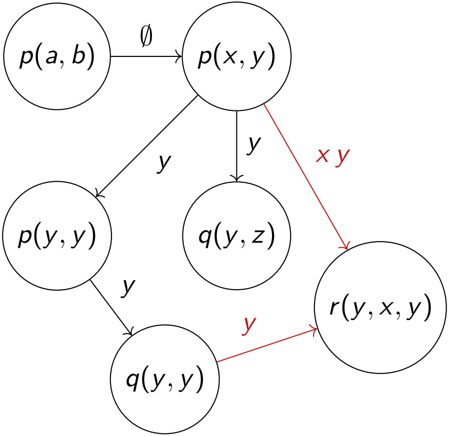

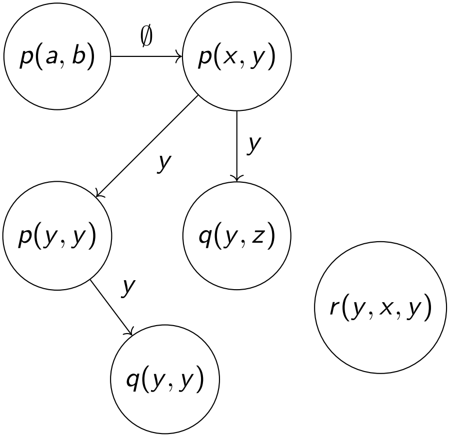

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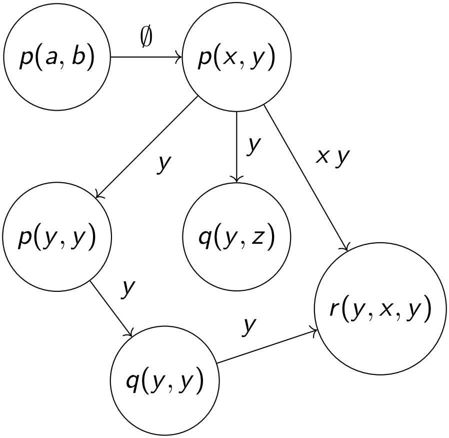

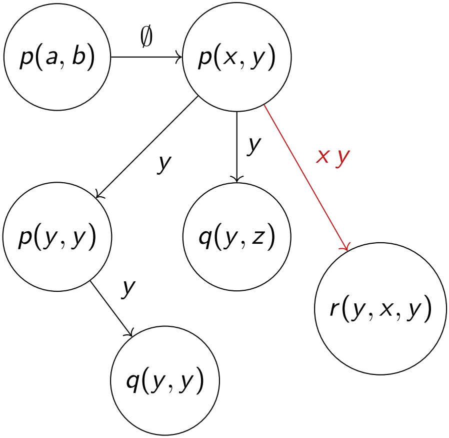

Derivation Graph Sets

$\mathcal{D} \xrightarrow{\delta} \mathcal{I}$ exhibits cycle-free derivation graph $\Rightarrow$ $\mathcal{I}$ has bounded treewidth.

1. A cycle-free derivation graph *corresponds to* a tree decomposition.
2. Width is bounded by $\max_{\rho \in \mathcal{R}}\{|\mathit{terms}(\mathcal{D})| + |\mathit{terms}(\mathit{head}(\rho))|\}$.

Cycle-free Derivation Graph Set (cdgs): If $\mathcal{D} \xrightarrow{\delta} \mathcal{I}$, then the derivation graph of δ is reducible to a cycle-free graph.

Weakly Cycle-free Derivation Graph Set (wcdgs): If $\mathcal{D} \xrightarrow{\delta} \mathcal{I}$, then there exists a derivation δ' such that the derivation graph of δ' is reducible to a cycle-free graph.

Theorem

1. **cdgs**, **wcdgs** $\subset$ **bts**.
2. *If* $\mathcal{R}$ *is* **cdgs** *or* **wcdgs**, *then query entailment is decidable.*

Greedy Derivations

Greedy Derivation (Thomazo et al. 2012) Frontier of any rule application is only mapped to elements which are (1) constants from $\mathcal{D}$, or (2) from the head of one single previous rule application.

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$\mathcal{D} = \{p(a, b)\}$
$\mathcal{R} = \{$

1. $p(x, y) \rightarrow \exists z.\ q(y, z),$
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$\mathcal{D}$ $\{p(a, b)\}$

$\mathcal{I}_1$ $\{p(a, b), q(b, x)\}$

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Greedy Derivations and Decidability

Greedy Bounded Treewidth Set (gbts): if $\mathcal{D} \xrightarrow{\delta} \mathcal{I}$, then δ is greedy.

Weakly Greedy Bounded Treewidth Set (wgbts): if $\mathcal{D} \xrightarrow{\delta} \mathcal{I}$, then there exists $\mathcal{D} \xrightarrow{\delta'} \mathcal{I}$ such that δ' is greedy.

Theorem

- (**w**)**gbts** = (**w**)**cdgs**.
- **gbts** $\subset$ **wgbts**.
- **cdgs** $\subset$ **wcdgs**.
- *All such rule sets have decidable BCQ entailment.*

Graph of Rule Dependencies

Dependency (Baget 2004): A rule ρ' depends on ρ, if ρ can "trigger" ρ'.

Graph of Rule Dependencies (Baget 2004): A directed graph showing all dependencies between rules.

$\mathcal{R} = \{$

1. $\rho_1 = p(x) \rightarrow \exists zw.q(x, z, w)$
2. $\rho_2 = r(y) \rightarrow \exists uv.s(y, u, v)$
3. $\rho_3 = p(x) \wedge r(y) \rightarrow \exists zwuv.q(x, z, w) \wedge s(y, u, v)$
4. $\rho_4 = q(x, z, w) \wedge s(y, u, v) \rightarrow \exists x'.t(x, z, y, u, x')$

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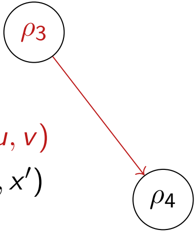

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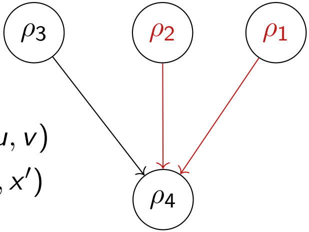

Graph of Rule Dependencies

Dependency (Baget 2004): A rule ρ' depends on ρ, if ρ can "trigger" ρ'.

Graph of Rule Dependencies (Baget 2004): A directed graph showing all dependencies between rules.

$\mathcal{R} = \{$

1. $\rho_1 = p(x) \to \exists zw.q(x,z,w)$
2. $\rho_2 = r(y) \to \exists uv.s(y,u,v)$
3. $\rho_3 = p(x) \wedge r(y) \to \exists zwuv.q(x,z,w) \wedge s(y,u,v)$
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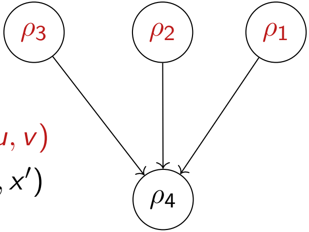

An Observation

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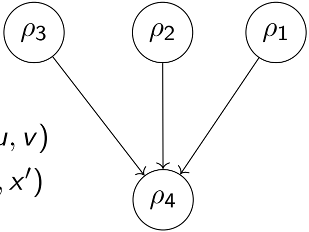

A Derivation: Let $\rho \in \{\rho_1, \rho_2, \rho_3\}$.

$$\mathcal{D}, \ldots, (\rho_i, h_i, \mathcal{I}_i), (\rho, h_{i+1}, \mathcal{I}_{i+1}), \ldots, (\rho_n, h_n, \mathcal{I}_n)$$

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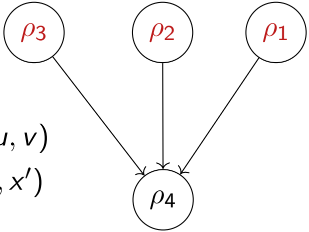

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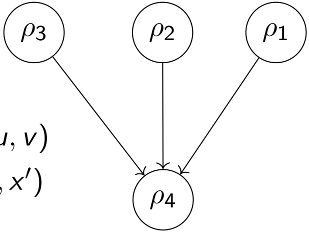

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The Observation: We can permute rule applications!

$$\mathcal{D}, \ldots, (\rho, h_{i+1}, \mathcal{I}_{i-1} \cup (\mathcal{I}_{i+1} \setminus \mathcal{I}_i)), (\rho_i, h_i, \mathcal{I}_{i+1}), \ldots, (\rho_n, h_n, \mathcal{I}_n)$$

Re-writing Derivations via Permutations

Lemma (Permutation Lemma)

Suppose we have a (greedy) derivation of the following form:

$$\delta := \mathcal{D}, \ldots, (\rho_i, h_i, \mathcal{I}_{i+1}), (\rho_{i+1}, h_{i+1}, \mathcal{I}_{i+2}), \ldots, (\rho_{n-1}, h_{n-1}, \mathcal{I}_n)$$

If ρ_{i+1} does not depend on ρ_i, then δ' is also a (greedy) derivation:

$$\delta' := \mathcal{D}, \ldots, (\rho_{i+1}, h_{i+1}, \mathcal{I}'_i), (\rho_i, h_i, \mathcal{I}_{i+2}), \ldots, (\rho_{n-1}, h_{n-1}, \mathcal{I}_n)$$

where $\mathcal{I}'_i = \mathcal{I}_i \cup (\mathcal{I}_{i+2} \setminus \mathcal{I}_{i+1})$.

Separating Example

$\mathcal{R} = \{$

1. $p(x) \rightarrow \exists zw.q(x, z, w)$
2. $r(y) \rightarrow \exists uv.s(y, u, v)$
3. $p(x) \wedge r(y) \rightarrow \exists zwuv.q(x, z, w) \wedge s(y, u, v)$
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$\mathcal{D}$ $\{p(a), r(b)\}$

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$\mathcal{D}$ $\{p(a), r(b)\}$

$\mathcal{I}_1$ $\{p(a), r(b), q(a, z, w)\}$

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$\mathcal{D}$ $\{p(a), r(b)\}$

$\mathcal{I}_1$ $\{p(a), r(b), q(a, z, w)\}$

$\mathcal{I}_2$ $\{p(a), r(b), q(a, z, w), s(b, u, v)\}$

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$\mathcal{I}_1$ $\{p(a), r(b), q(a, z, w)\}$

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$\mathcal{I}_3$ $\{p(a), r(b), q(a, z, w), s(b, u, v), t(a, z, b, u, x')\}$

$\mathcal{R}$ is not **gbts**, but is $\mathcal{R}$ **wgbts**?!

Showing $\mathcal{R}$ is **wgbts**

Idea 1: Use 3 to simulate 1 and 2.

$\mathcal{R} = \{$

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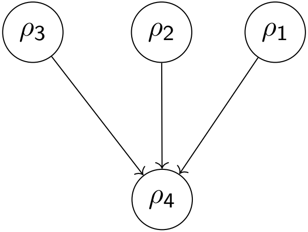

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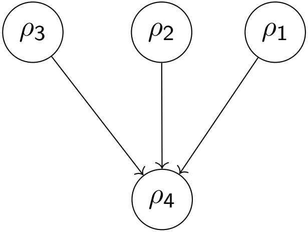

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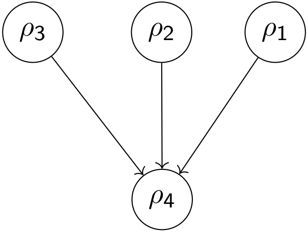

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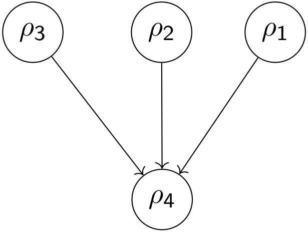

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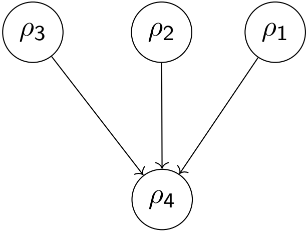


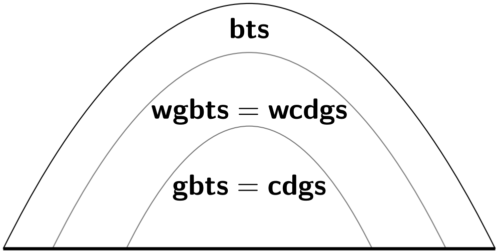
Idea 2: Permutation Lemma $\rightsquigarrow$

(i) Permute instances of 1 or 2 backward to instances of 2 or 1 (resp.)

(ii) Replace 1,2 or 2,1 instances with 3.

Summary and Future Work

Our Results:



Open Questions:

1. Are **gbts** and **wgbts** recognizable?
2. What is the complexity of BCQ entailment in (**w**)**gbts**?
3. Can other reduction operations generalize (**w**)**cdgs**?