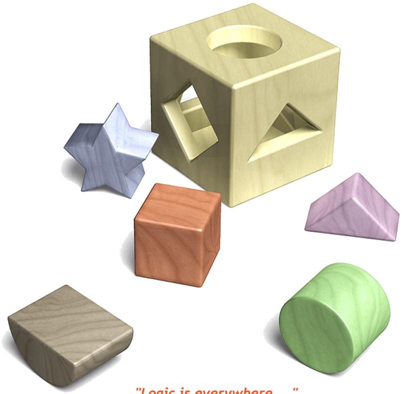
SAT Solving – Parallel Search

Steffen Hölldobler and Norbert Manthey
International Center for Computational Logic
Technische Universität Dresden
Germany

- **Parallel Approaches**
- **Abstract Description**
- **High-Level**
- **Problems**



"Logic is everywhere ..."

Parallelization – Warm Up

- **If an algorithm has three parts that consume 80 %, 10 % and 10 %**

Parallelization – Warm Up

- **If an algorithm has three parts that consume 80 %, 10 % and 10 %**
- **Which task should be parallelized?**

Parallelization – Warm Up

- **If an algorithm has three parts that consume 80 %, 10 % and 10 %**
- **Which task should be parallelized?**
- **What is the ideal speedup for two cores?**

Parallelization – Warm Up

- **Assume two solvers S_1 and S_2 solve the formula F independently**
- **S_1 learns C, S_2 learns D**

Parallelization – Warm Up

- **Assume two solvers S_1 and S_2 solve the formula F independently**

- **S_1 learns C, S_2 learns D**

- **If S_1 receives D, is satisfiability preserved?**

Parallelization – Warm Up

- Assume two solvers S_1 and S_2 solve the formula F independently
- S_1 learns C, S_2 learns D
- If S_1 receives D, is satisfiability preserved?
- If S_1 receives D, is equivalence preserved?

Parallelization – Warm Up

- **Run unit propagation on**
  $F = (\neg e \lor f) \land (\neg a \lor e) \land (\neg c \lor d) \land (\neg b \lor c) \land (\neg a \lor b) \land a$

Parallelization – Warm Up

- **Run unit propagation on**
  $F = (\neg e \lor f) \land (\neg a \lor e) \land (\neg c \lor d) \land (\neg b \lor c) \land (\neg a \lor b) \land a$

- **How would you parallelize?**

Parallelization – Warm Up

- **Run unit propagation on**
  $F = (\neg e \lor f) \land (\neg a \lor e) \land (\neg c \lor d) \land (\neg b \lor c) \land (\neg a \lor b) \land a$

- **How would you parallelize?**

- **Can the steps be run in parallel?**

Parallelization – Warm Up

- **Run unit propagation on**
  $F = (\neg e \vee f) \wedge (\neg a \vee e) \wedge (\neg c \vee d) \wedge (\neg b \vee c) \wedge (\neg a \vee b) \wedge a$
- **How would you parallelize?**
- **Can the steps be run in parallel?**
- **Run unit propagation on**
  $F = (\neg e \vee f) \wedge (\neg d \vee e) \wedge (\neg c \vee d) \wedge (\neg b \vee c) \wedge (\neg a \vee b) \wedge a$

Parallelization – Warm Up

- **Run unit propagation on**
  $F = (\neg e \vee f) \wedge (\neg a \vee e) \wedge (\neg c \vee d) \wedge (\neg b \vee c) \wedge (\neg a \vee b) \wedge a$
- **How would you parallelize?**
- **Can the steps be run in parallel?**
- **Run unit propagation on**
  $F = (\neg e \vee f) \wedge (\neg d \vee e) \wedge (\neg c \vee d) \wedge (\neg b \vee c) \wedge (\neg a \vee b) \wedge a$
- **From a complexity point of view, this is an open question!**

Parallelization – Revision

- A sequential algorithm
  - requires time t_1 seconds
- A parallel algorithm
  - utilizes p computation units (cores)
  - requires time t_p seconds

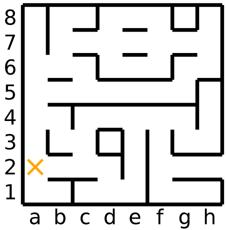
- The speedup $S_p = \frac{t_1}{t_p}$
  - A speedup S_p is superlinear, if $S_p > p$
- The efficiency $E_p = \frac{S_p}{p}$
- An algorithm is called scalable,

  if it can solve a given problem faster when additional resources are added

Abstract Visualization

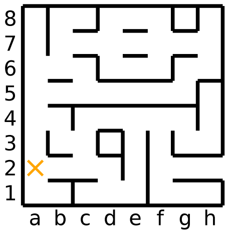
Finding an Exit in a Maze

- **Some rules**
  - **starting point is located in the left column**
  - **exit is on the right side (if there exists one)**
  - **search decisions can be done only when moving right**
  - **when moving left, use backtracking**



Finding an Exit in a Maze

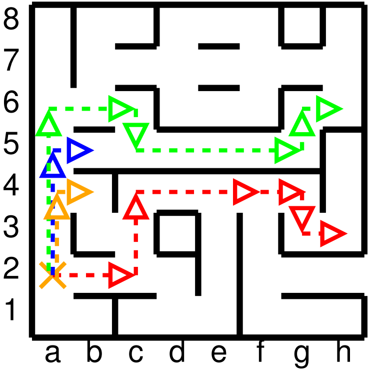
- **Some rules**
  - **starting point is located in the left column**
  - **exit is on the right side (if there exists one)**
  - **search decisions can be done only when moving right**
  - **when moving left, use backtracking**



- **Parallel approaches: multiple searches, search space splitting**

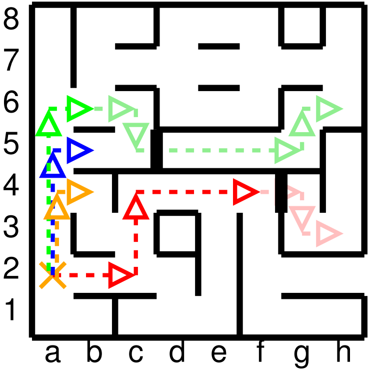
Solve With Multiple Solvers

- Use a solver per computing resource
  - Use different heuristics
  - Solvers work independently



Solve With Multiple Solvers

- Learned clauses can be shared
  - All solvers work on the same formula
  - No simplification involved



Solve With Multiple Solvers

- **Arising questions:**
  - **How scalable is the presented approach?**

Solve With Multiple Solvers

- **Arising questions:**
  - **How scalable is the presented approach?**
  - **What influences scalability?**

Solve With Multiple Solvers

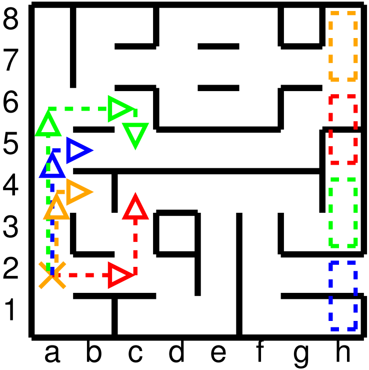
- **Arising questions:**
  - **How scalable is the presented approach?**
  - **What influences scalability?**
  - **How long is clause sharing valid wrt simplification?**

Solve With Multiple Solvers

- **Arising questions:**
  - **How scalable is the presented approach?**
  - **What influences scalability?**
  - **How long is clause sharing valid wrt simplification?**
  - **Is there a good alternative?**

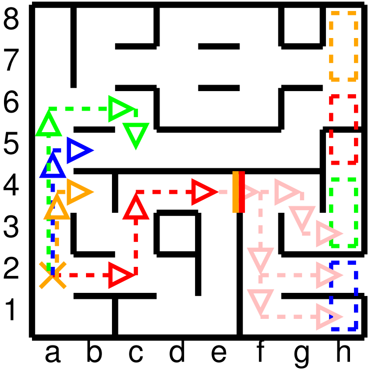
Partition Search Space

- Create a partition per computing resource
  - Assign a solver to each partition
  - Solvers work independently



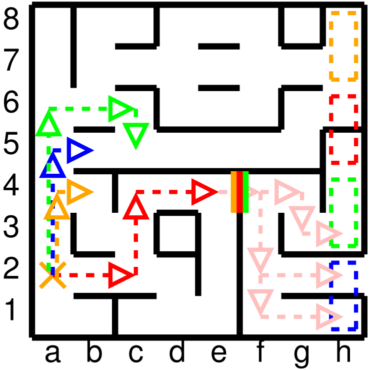
Partition Search Space

- Learned clauses can be shared



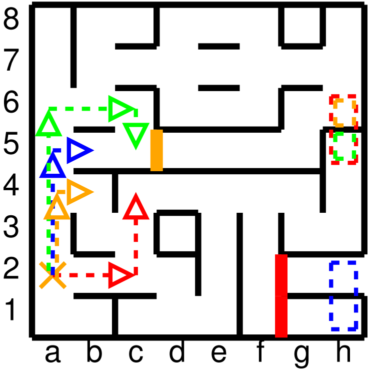
Partition Search Space

- Learned clauses can be shared
  - with respect to the partition
  - carefully



Partition Search Space

- Partitions can be re-partitioned
  - Ensure load balancing and applies many resources to hard partitions
  - Possible to use learned clauses of parent partition



High Level Parallelization Approaches

Parallel Portfolio Solvers

Search Space Partitioning Solvers

Parallel Portfolio Solvers

- **Solve a formula F with n resources**
- **Idea: solve F with multiple solvers**

Parallel Portfolio Solvers

- Solve a formula F with n resources
- Idea: solve F with multiple solvers
  - With different configurations

Parallel Portfolio Solvers

- Solve a formula F with n resources
- Idea: solve F with multiple solvers
  - With different configurations
  - With knowledge sharing (learned clauses)

Parallel Portfolio Solvers

- Solve a formula F with n resources
- Idea: solve F with multiple solvers
  - With different configurations
  - With knowledge sharing (learned clauses)

- Drawbacks:
  - Known to not scale well
  - A small set of configurations is already robust
  - Scalability is independent of formula size

Parallel Portfolio Solvers – Sharing

- Given two solvers S_1 and S_2, and let S_1 learn a clause C
- Let F^1 and F^2 be the working formulas of S_1 and S_2, respectively
- When is S_2 allowed to receive C

Parallel Portfolio Solvers – Sharing

- Given two solvers S_1 and S_2, and let S_1 learn a clause C
- Let F^1 and F^2 be the working formulas of S_1 and S_2, respectively
- When is S_2 allowed to receive C
  - If $F^2 \equiv F^2 \wedge C$
  - If $F^2 \equiv_{\mathsf{SAT}} F^2 \wedge C$

Parallel Portfolio Solvers – Sharing

- Given two solvers S_1 and S_2, and let S_1 learn a clause C
- Let F^1 and F^2 be the working formulas of S_1 and S_2, respectively
- When is S_2 allowed to receive C
  - If $F^2 \equiv F^2 \wedge C$
  - If $F^2 \equiv_{SAT} F^2 \wedge C$
  - The check is done implicitely, as it is too expensive
    - No simplification, then all clauses are entailed
    - Only clause elimination / model increasing techniques, then sharing preserves equisatisfiability

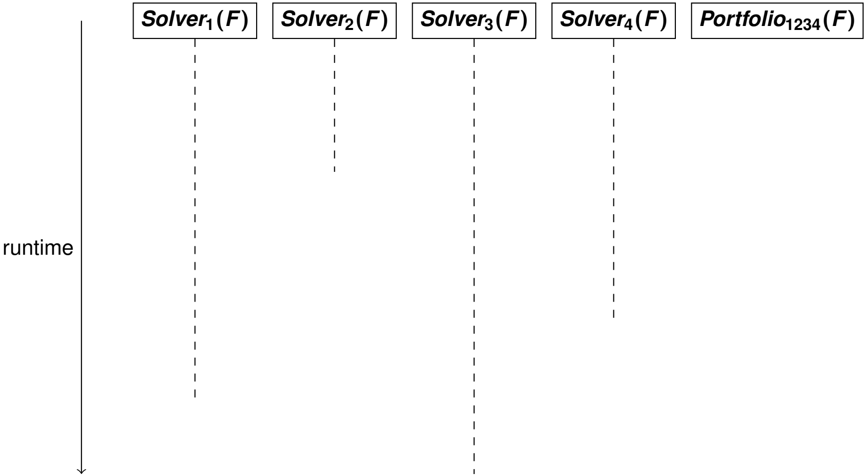
Parallel Portfolio Solvers – Sharing

- Given two solvers S_1 and S_2, and let S_1 learn a clause C
- Let F^1 and F^2 be the working formulas of S_1 and S_2, respectively
- When is S_2 allowed to receive C
  - If $F^2 \equiv F^2 \land C$
  - If $F^2 \equiv_{\text{SAT}} F^2 \land C$
  - The check is done implicitely, as it is too expensive
    - No simplification, then all clauses are entailed
    - Only clause elimination / model increasing techniques, then sharing preserves equisatisfiability
  - Addition of redundant, but not entailed, clauses
    - Extra care, otherwise:
    - $F^1 = x$, and S_2 applies simplification: $F^2 = x \leadsto_{\text{modelInc}} F^2 = \emptyset \leadsto_{\text{modelDec}} F^2 = \overline{x}$
    - Solver S_2 reveices (x) from S_1: $F^2 = \overline{x} \land x$
    - Satisfiable formula is found to be unsatisfiable

Parallel Portfolio Solvers – Sharing

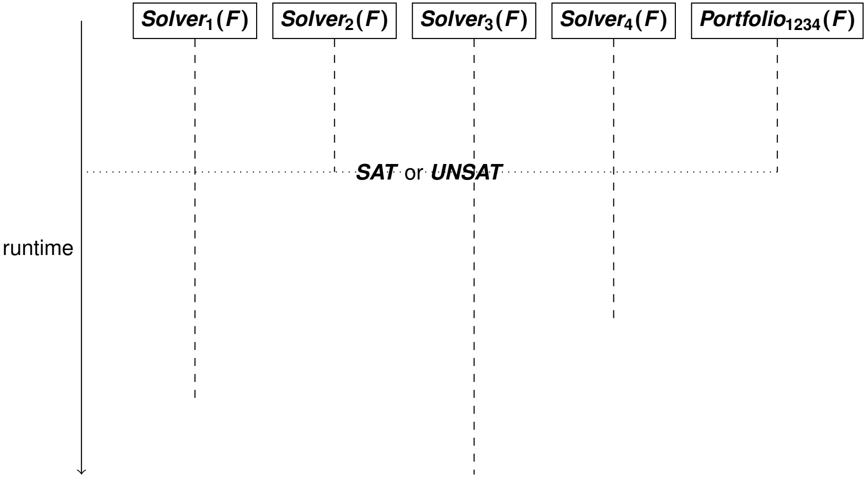
- Given two solvers S_1 and S_2, and let S_1 learn a clause C
- Let F^1 and F^2 be the working formulas of S_1 and S_2, respectively
- When is S_2 allowed to receive C
  - If $F^2 \equiv F^2 \wedge C$
  - If $F^2 \equiv_{SAT} F^2 \wedge C$
  - The check is done implicitely, as it is too expensive
    - No simplification, then all clauses are entailed
    - Only clause elimination / model increasing techniques, then sharing preserves equisatisfiability
  - Addition of redundant, but not entailed, clauses
    - Do not receive clauses, if a model decreasing has been used

Solving SAT in parallel with the Portfolio approach



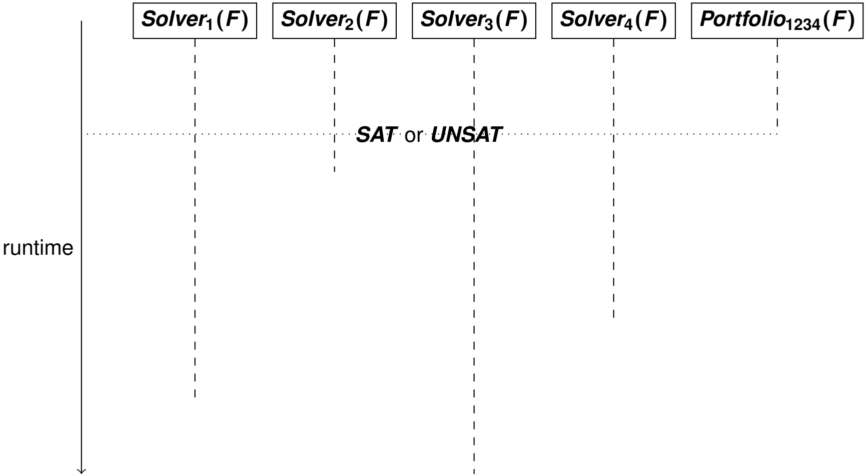
- **Different SAT solvers compete**

Solving SAT in parallel with the Portfolio approach



- **The portfolio of these solvers requires the smallest run time**

Solving SAT in parallel with the Portfolio approach



- **By adding communication among the solvers, the performance can be improved**

High Level Parallelization Approaches

Parallel Portfolio Solvers

Search Space Partitioning Solvers

Search Space Partitioning

- Partition search space of formula F into sub spaces:
  - For some $r > 0$ create r "child"-formulas F^i, $0 < i \leq r$, such that
    - their disjunction is equal to the initial formula $F \equiv \bigvee F^i$
    - a partition constraint K^i in CNF is added, $F^i := F \wedge K^i$

Search Space Partitioning

- Partition search space of formula F into sub spaces:
  - For some $r > 0$ create r “child”-formulas F^i, $0 < i \leq r$, such that
    - their disjunction is equal to the initial formula $F \equiv \bigvee F^i$
    - a partition constraint K^i in CNF is added, $F^i := F \wedge K^i$
  - To obtain a partition, additionally ensure
    - that the child-formulas represent disjoint search spaces $F^i \wedge F^j \equiv \perp$, for all $0 \leq i < j \leq r$.

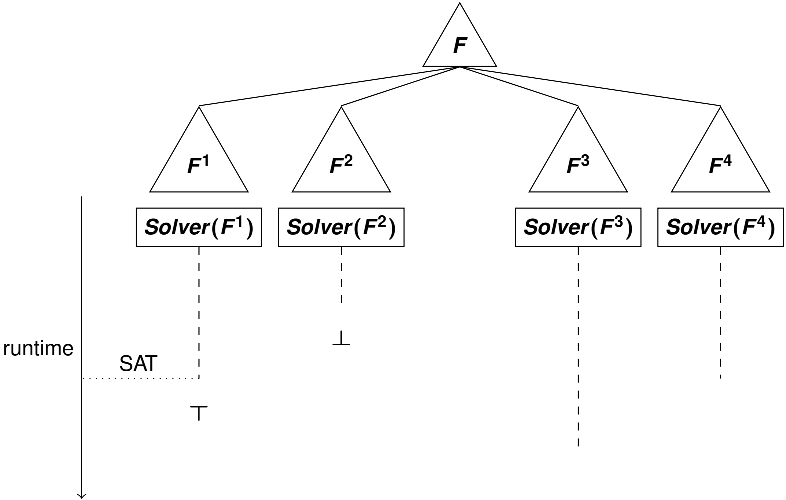
Search Space Partitioning

- Partition search space of formula F into sub spaces:
  - For some $r > 0$ create r “child”-formulas F^i, $0 < i \leq r$, such that
    - their disjunction is equal to the initial formula $F \equiv \bigvee F^i$
    - a partition constraint K^i in CNF is added, $F^i := F \wedge K^i$
  - To obtain a partition, additionally ensure
    - that the child-formulas represent disjoint search spaces $F^i \wedge F^j \equiv \perp$, for all $0 \leq i < j \leq r$.
  - Solve each child-formula with a sequential solver
  - If a solver proofed unsatisfiability of a sub formula
    - assign a new child formula

Search Space Partitioning

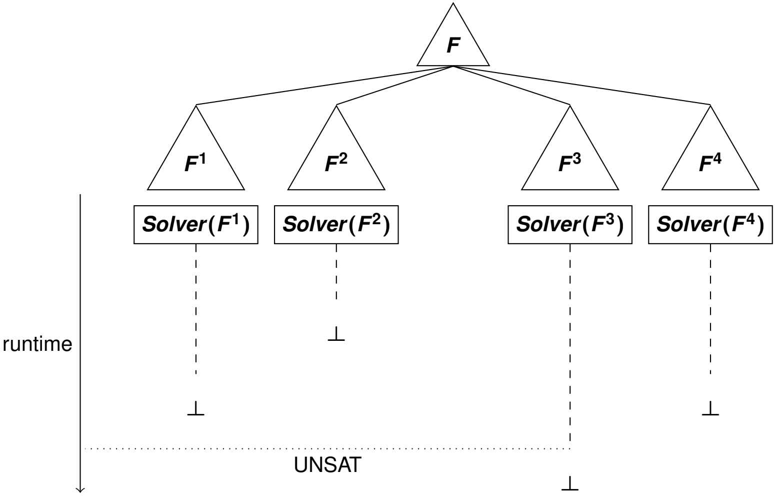
- Partition search space of formula F into sub spaces:
  - For some $r > 0$ create r "child"-formulas F^i, $0 < i \leq r$, such that
    - their disjunction is equal to the initial formula $F \equiv \bigvee F^i$
    - a partition constraint K^i in CNF is added, $F^i := F \wedge K^i$
  - To obtain a partition, additionally ensure
    - that the child-formulas represent disjoint search spaces $F^i \wedge F^j \equiv \perp$, for all $0 \leq i < j \leq r$.
  - Solve each child-formula with a sequential solver
  - If a solver proofed unsatisfiability of a sub formula
    - assign a new child formula
  - Load-balancing is usually handled by providing sufficiently many child formulas
  - Can scale with the number of created child formulas, if partitioning works

(Plain) Search Space Partitioning



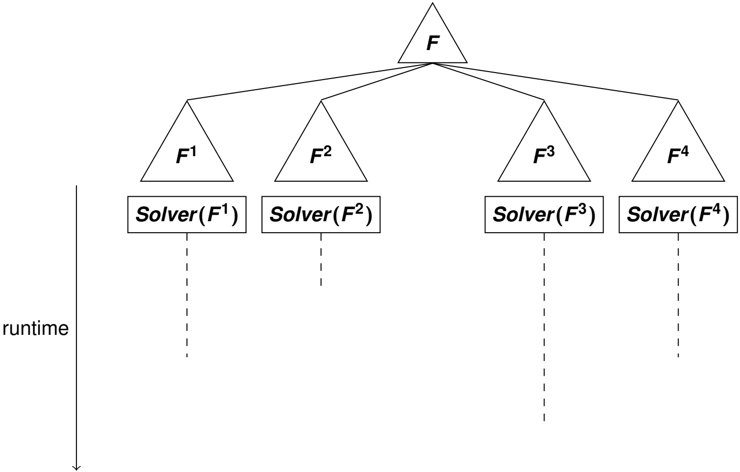
- **finds models as fast as the fastest solver**

(Plain) Search Space Partitioning



- **proofs unsatisfiability as slow as the slowest solver**

(Plain) Search Space Partitioning



- **not ensured:**
  $$max(t_{Solver}(F^1), t_{Solver}(F^2), t_{Solver}(F^3), t_{Solver}(F^4)) \leq (t_{Solver}(F))$$

Iterative Search Space Partitioning

- Partition search space of formula F into sub spaces:
  - For some $r > 0$ create r “child”-formulas F^i, $0 \leq i \leq r$, such that
    - $F \equiv \bigvee F^i$
    - $F^i \wedge F^j \equiv \perp$, for all $0 \leq i < j \leq r$.
  - Solve all formulas with a sequential solver (not only child formulas)
  - If a solver proofed unsatisfiability of a sub formula, assign a new child formula
    - assign a new child formula
    - or by recursively applying the partitioning procedure to child formulas

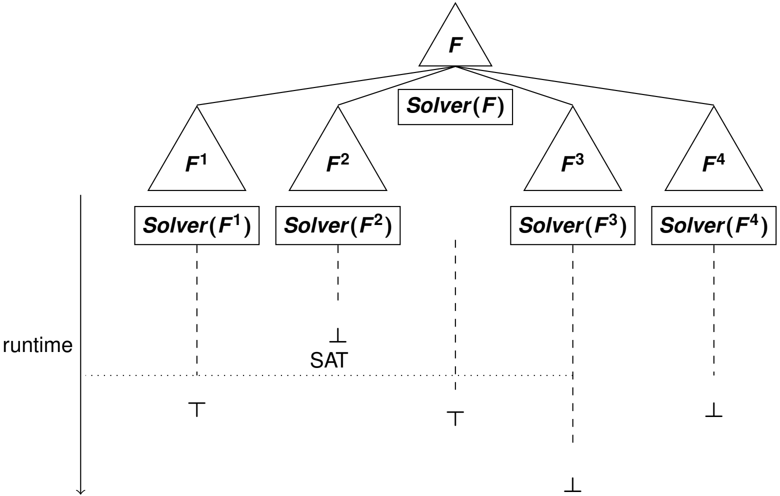
Iterative Search Space Partitioning

- Partition search space of formula F into sub spaces:
  - For some $r > 0$ create r "child"-formulas F^i, $0 \leq i \leq r$, such that
    - $F \equiv \bigvee F^i$
    - $F^i \wedge F^j \equiv \bot$, for all $0 \leq i < j \leq r$.
  - Solve all formulas with a sequential solver (not only child formulas)
  - If a solver proofed unsatisfiability of a sub formula, assign a new child formula
    - assign a new child formula
    - or by recursively applying the partitioning procedure to child formulas
  - Creates a breadth first search in the search space

Iterative Search Space Partitioning

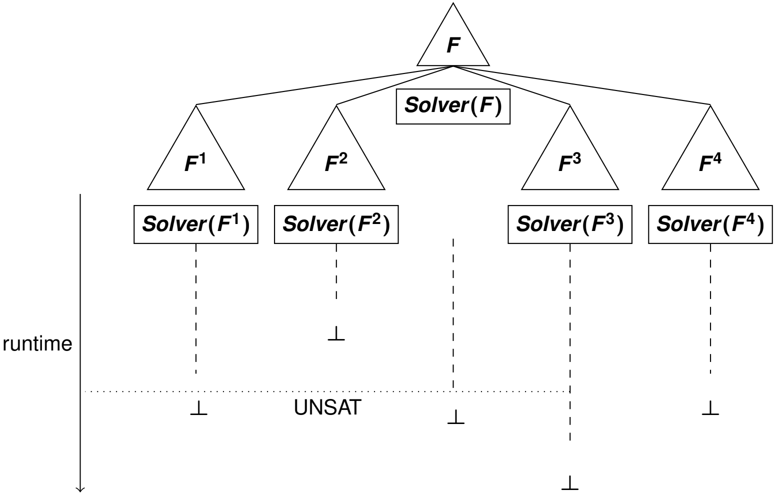
- Partition search space of formula F into sub spaces:
  - For some $r > 0$ create r "child"-formulas F^i, $0 \leq i \leq r$, such that
    - $F \equiv \bigvee F^i$
    - $F^i \wedge F^j \equiv \perp$, for all $0 \leq i < j \leq r$.
  - Solve all formulas with a sequential solver (not only child formulas)
  - If a solver proofed unsatisfiability of a sub formula, assign a new child formula
    - assign a new child formula
    - or by recursively applying the partitioning procedure to child formulas
  - Creates a breadth first search in the search space
  - Can assign new child-formulas to new resources by iterative partitioning

Iterative Search Space Partitioning



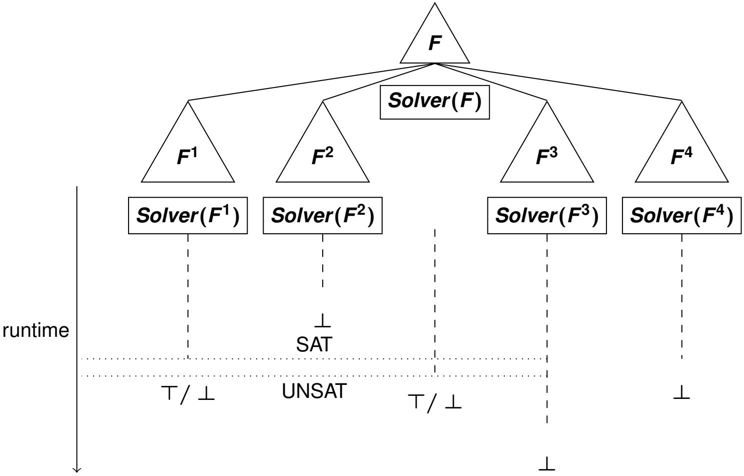
- **finds models as fast as the fastest solver**

Iterative Search Space Partitioning



- **proofs unsatisfiability as fast as the slowest "necessary" solver**

Iterative Search Space Partitioning



- **by iteratively partitioning the search space, new child formulas become more constrained**

Iterative Partitioning – Dependencies

- Solve formula F
- Create a tree
  - Create partitioning constraints K^i with $1 \leq i \leq r$, for some r
  - $F \equiv \bigvee_{1 \leq i \leq k} (F \land K^i)$
  - $K^i \land K^j \equiv \bot$ for all $1 \leq i < j \leq k$

- Definition
  - A clause C depends on a path p, if p is the longest path of all clauses that participated in the derivation of C.

Iterative Partitioning – Dependencies

$$F^p := ((x_1 \lor x_2 \lor x_5) \land (x_3 \lor x_4) \land (\overline{x_2}, x_6, x_1) \land (\overline{x_2} \lor \overline{x_6}))$$

$(\overline{x_1})$ $\qquad\qquad$ (x_1)

$$F^{p1} := ((x_2 \lor x_5) \land (x_3 \lor x_4) \land (\overline{x_2} \lor x_6) \land \ldots) \qquad F^{p2} := ((x_3 \lor x_4) \land \ldots)$$

Iterative Partitioning – Dependencies

$$F^{p} := ((x_1 \vee x_2 \vee x_5) \wedge (x_3 \vee x_4) \wedge (\overline{x_2}, x_6, x_1) \wedge (\overline{x_2} \vee \overline{x_6}))$$

$(\overline{x_1})$ $\qquad$ (x_1)

$$F^{p1} := ((x_2 \vee x_5) \wedge (x_3 \vee x_4) \wedge (\overline{x_2} \vee x_6) \wedge \ldots) \qquad F^{p2} := ((x_3 \vee x_4) \wedge \ldots)$$

- The clauses $(x_2 \vee x_5)$ and $(\overline{x_2} \vee x_6)$ depend on the partitioning constraint

Iterative Partitioning – Dependencies

$$F^p := ((x_1 \vee x_2 \vee x_5) \wedge (x_3 \vee x_4) \wedge (\overline{x_2}, x_6, x_1) \wedge (\overline{x_2} \vee \overline{x_6}))$$

$(\overline{x_1})$ (x_1)

$$F^{p1} := ((x_2 \vee x_5) \wedge (x_3 \vee x_4) \wedge (\overline{x_2} \vee x_6) \wedge \ldots) \qquad F^{p2} := ((x_3 \vee x_4) \wedge \ldots)$$

- **The clauses $(x_2 \vee x_5)$ and $(\overline{x_2} \vee x_6)$ depend on the partitioning constraint**
- **Their label has to be adapted accordingly!**

Iterative Partitioning – A Abstract Example

- Solve formula F
- Create a tree
  - Create partitioning constraints K^i with $1 \leq i \leq r$, for some r
  - $F \equiv \bigvee_{1 \leq i \leq k}(F \wedge K^i)$
  - $K^i \wedge K^j \equiv \perp$ for all $1 \leq i < j \leq k$
    - e.g. $K^1 = x \wedge y$, $K^2 = ((\neg x \vee \neg y) \wedge c)$ and $K^3 = ((\neg x \vee \neg y) \wedge \neg c)$
  - Label each node with its path to the root node,
    e.g. $F^{132} = F \wedge K^1 \wedge K^{13} \wedge K^{132}$

Iterative Partitioning – A Abstract Example

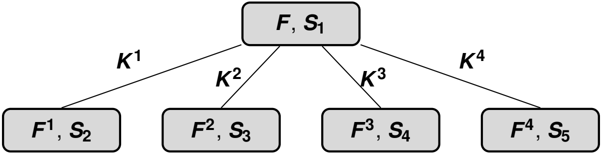
- Solve formula F
- Create a tree
  - Create partitioning constraints K^i with $1 \leq i \leq r$, for some r
  - $F \equiv \bigvee_{1 \leq i \leq k} (F \wedge K^i)$
  - $K^i \wedge K^j \equiv \bot$ for all $1 \leq i < j \leq k$
    - e.g. $K^1 = x \wedge y$, $K^2 = ((\neg x \vee \neg y) \wedge c)$ and $K^3 = ((\neg x \vee \neg y) \wedge \neg c)$
  - Label each node with its path to the root node,
    e.g. $F^{132} = F \wedge K^1 \wedge K^{13} \wedge K^{132}$
- Have one solver for each core, assign a node to each solver

Iterative Partitioning – A Abstract Example

- Solve formula F
- Create a tree
  - Create partitioning constraints K^i with $1 \leq i \leq r$, for some r
  - $F \equiv \bigvee_{1 \leq i \leq k}(F \wedge K^i)$
  - $K^i \wedge K^j \equiv \perp$ for all $1 \leq i < j \leq k$
    - e.g. $K^1 = x \wedge y$, $K^2 = ((\neg x \vee \neg y) \wedge c)$ and $K^3 = ((\neg x \vee \neg y) \wedge \neg c)$
  - Label each node with its path to the root node,
    e.g. $F^{132} = F \wedge K^1 \wedge K^{13} \wedge K^{132}$
- Have one solver for each core, assign a node to each solver
- Partition nodes recursively if resources become available again

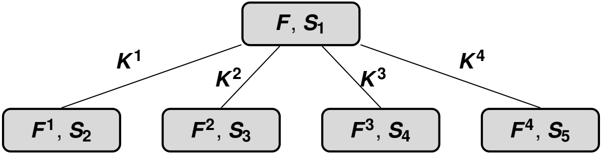
Iterative Partitioning – A Abstract Example

- **Created 4 nodes with their partitioning constraints**
- **Assign all 5 solvers S_1 to S_5 to nodes**

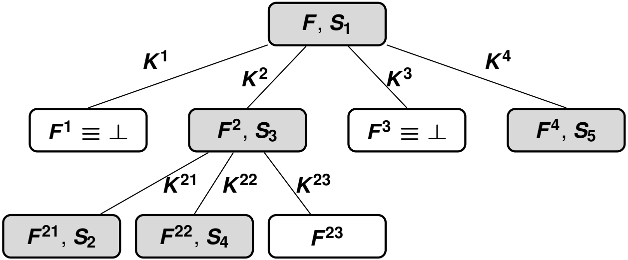


Iterative Partitioning – A Abstract Example

- Solver S_2 and S_4 find their formula to be unsatisfiable
- Partition F^2, and assign the solvers again

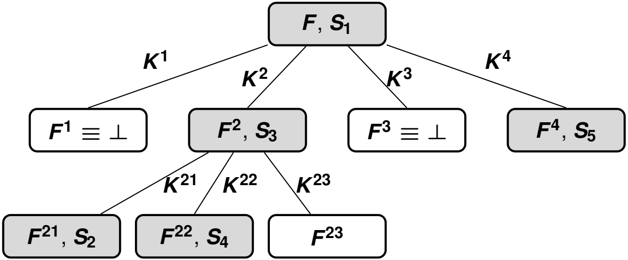


Iterative Partitioning – A Abstract Example

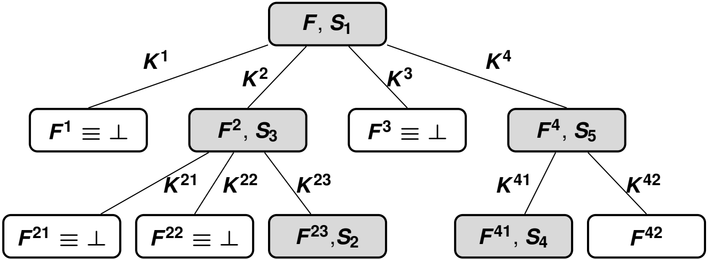


Iterative Partitioning – A Abstract Example

- Solver S_2 and S_4 find their formula to be unsatisfiable
- Assign S_2 to F^{21}, partition F^4, and assign S_4 to F^{41}

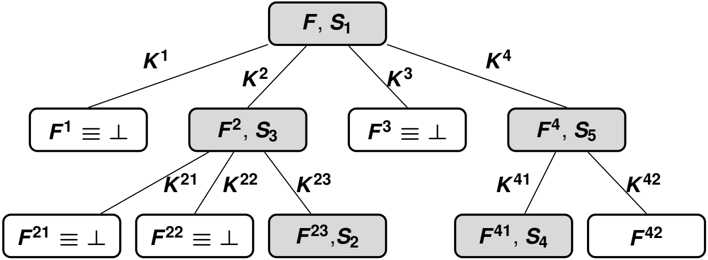


Iterative Partitioning – A Abstract Example



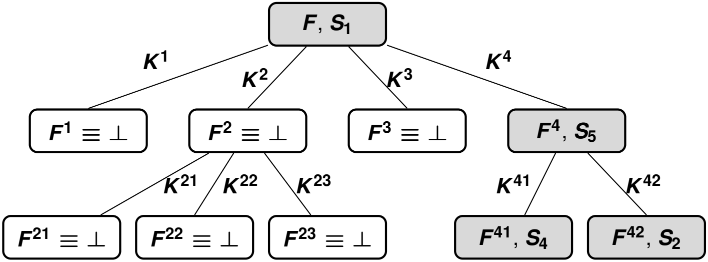
Iterative Partitioning – A Abstract Example

- Solver S_2 finds F^{23} to be unsatisfiable
- F^2 has to be unsatisfiable as well



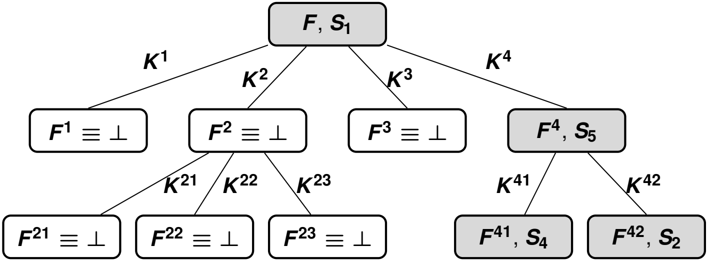
Iterative Partitioning – A Abstract Example

- Solver S_2 finds F^{23} to be unsatisfiable
- F^2 has to be unsatisfiable as well



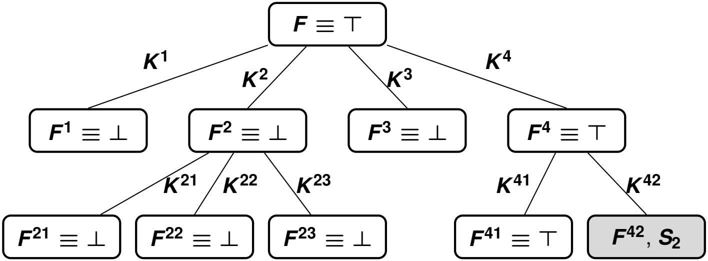
Iterative Partitioning – A Abstract Example

- Solver S_4 finds F^{41} to be satisfiable
- Then F^4 and F are satisfiable as well



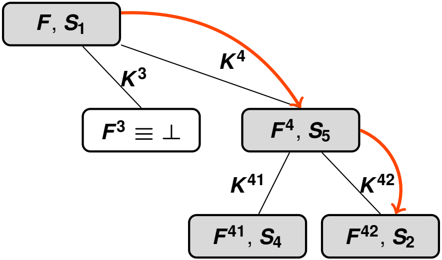
Iterative Partitioning – A Abstract Example

- Solver S_4 finds F^{41} to be satisfiable
- Then F^4 and F are satisfiable as well



Iterative Partitioning – Downward Sharing

- **Solver S_1 learns clause C**
- **Downward clause sharing is safe, $F \models C$, then $F \land K^i \models C$**
- **Assumption: no simplification**



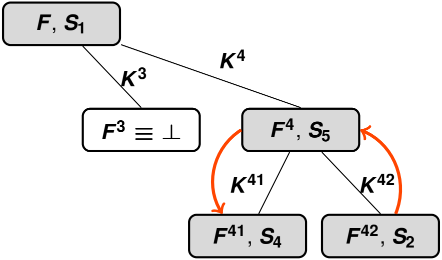
$$F \models C$$

$$F^4 = F \land K^4 \models C$$

$$F^{41} = F \land K^4 \land K^{41} \models C$$

Iterative Partitioning – Upward Sharing

- Solver S_2 learns clause C, $F \wedge K^4 \wedge K^{42} \models C$
- Suppose C depends only on F and K^4
- Upward clause sharing to F^4 is safe
- Store dependency level for each clause
- Assumption: no simplification

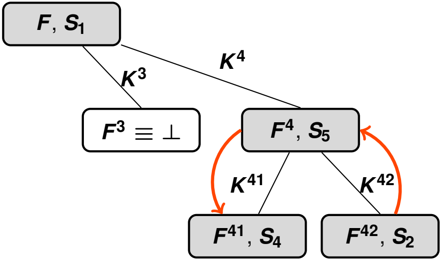


$$F^4 = F \wedge K^4 \models C$$

$$F^{41} = F \wedge K^4 \wedge K^{41} \models C$$

Iterative Partitioning – Abort Redundant

- Solver S_2 learns empty clause $\bot$, $F \wedge K^4 \wedge K^{42} \models \bot$
- Suppose the empty clause depends only on F and K^4
- Upward clause sharing to F^4 is safe
- Abort all solvers below F^4 (S_2, S_4 and S_5)



$F \models \bot$

$F^4 = F \wedge K^4 \models \bot$

$F^{41} = F \wedge K^4 \wedge K^{41} \models \bot$

Partition Tree With Shared Clauses

$$F = ((x_1 \vee x_2 \vee x_3) \wedge (\overline{x_3} \vee \overline{x_2})$$
$$\wedge(\overline{x_2} \vee x_4 \vee \overline{x_1}) \wedge (\overline{x_2} \vee \overline{x_4} \vee \overline{x_1})$$
$$\wedge(x_4 \vee x_2 \vee x_5 \vee \overline{x_7}) \wedge (\overline{x_4} \vee x_2 \vee \overline{x_5}) \wedge (x_7 \vee x_8))$$

(x_7) $(\overline{x_7})$

$$F^1 = (\ldots \wedge (\overline{x_2} \vee \overline{x_4} \vee \overline{x_1}) \wedge (x_4 \vee x_2 \vee x_5) \wedge (x_4 \vee x_2 \vee \overline{x_5}))$$

F^2

(x_1) $(\overline{x_1})$

$$((\overline{x_3} \vee \overline{x_2}) \wedge (\overline{x_2} \vee x_4)$$
$$\wedge(\overline{x_2} \vee \overline{x_4}) \wedge \ldots)$$

$$((x_2 \vee x_3) \wedge (\overline{x_3} \vee \overline{x_2}) \wedge \ldots)$$

(x_3) $(\overline{x_3})$

$$F^{121} = ((\overline{x_2}) \wedge (x_4 \vee x_2 \vee x_5)$$
$$\wedge(x_4 \vee x_2 \vee \overline{x_5}))$$

$$((x_2) \wedge (x_4 \vee x_2 \vee x_5)$$
$$\wedge(x_4 \vee x_2 \vee \overline{x_5}))$$

- $D = (x_4 \vee x_2)$ is learned by $(x_4 \vee x_2 \vee x_5) \otimes (x_4 \vee x_2 \vee \overline{x_5})$ in formula F^{121}
- D depends on the partition constraint x_7

Partition Tree With Shared Clauses

$$F = ((x_1 \vee x_2 \vee x_3) \wedge (\overline{x_3} \vee \overline{x_2})$$
$$\wedge(\overline{x_2} \vee x_4 \vee \overline{x_1}) \wedge (\overline{x_2} \vee \overline{x_4} \vee \overline{x_1})$$
$$\wedge(x_4 \vee x_2 \vee x_5 \vee \overline{x_7}) \wedge (\overline{x_4} \vee x_2 \vee \overline{x_5}) \wedge (x_7 \vee x_8))$$

(x_7) $\qquad$ $(\overline{x_7})$

$$F^1 = (\ldots \wedge (\overline{x_2} \vee \overline{x_4} \vee \overline{x_1}) \wedge (x_4 \vee x_2 \vee x_5) \wedge (x_4 \vee x_2 \vee \overline{x_5}))$$ $\qquad$ F^2

(x_1) $\qquad$ $(\overline{x_1})$

$$((\overline{x_3} \vee \overline{x_2}) \wedge (\overline{x_2} \vee x_4)$$
$$\wedge(\overline{x_2} \vee \overline{x_4}) \wedge \ldots)$$

$$((x_2 \vee x_3) \wedge (\overline{x_3} \vee \overline{x_2}) \wedge \ldots)$$

(x_3) $\qquad$ $(\overline{x_3})$

$$F^{121} = ((\overline{x_2}) \wedge (x_4 \vee x_2 \vee x_5)$$
$$\wedge(x_4 \vee x_2 \vee \overline{x_5}))$$

$$((x_2) \wedge (x_4 \vee x_2 \vee x_5)$$
$$\wedge(x_4 \vee x_2 \vee \overline{x_5}))$$

- $D = (x_4 \vee x_2)$ is learned by $(x_4 \vee x_2 \vee x_5) \otimes (x_4 \vee x_2 \vee \overline{x_5})$ in formula F^{121}
- D depends on the partition constraint x_7
- Hence, D can be shared in the subtree of F^1

Not Discussed Here

- **Sharing and Simplification**
- **Low-Level Parallelization**
- **Parallel Simplification**