Extracting Confident General Concept Inclusions from Finite Interpretations

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Overview

1. Motivation and Introduction
2. Formal Concept Analysis
3. Description Logics
4. Confident General Concept Inclusions of Finite Interpretations

Automation is a Good Idea

Goal (of AI)

Let computers do what they can do (and leave all the rest to the humans.)

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Represent knowledge in a way suitable for computers,

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Represent knowledge in a way suitable for computers, i. e. as a *description logics ontology*.

Description Logics Ontologies

Ontologies contain assertional knowledge and terminological knowledge.

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Example ($\mathcal{EL}^{\bot}$-Ontology)

$(\mathcal{T}, \mathcal{A})$ is an ontology, where

$$\begin{aligned}\mathcal{T} &= \{\, \mathsf{Cat} \sqsubseteq \mathsf{Animal} \sqcap \exists \mathsf{hunts}.\mathsf{Mouse}, \\ &\qquad \mathsf{Cat} \sqcap \mathsf{Mouse} \sqsubseteq \bot \,\} \\ \mathcal{A} &= \{\, \mathsf{Cat}(\mathsf{Tom}), \mathsf{Mouse}(\mathsf{Jerry}), \mathsf{hunts}(\mathsf{Tom}, \mathsf{Jerry}) \,\}\end{aligned}$$

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Definition

Terminological axioms of the form $C \sqsubseteq D$ are called *general concept inclusions* (GCIs.)

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Automatically construct ontologies from unstructured data

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Semi-Automatically construct the terminological part of ontologies from unstructured data

Unstructured Data

Question

What is unstructured data?

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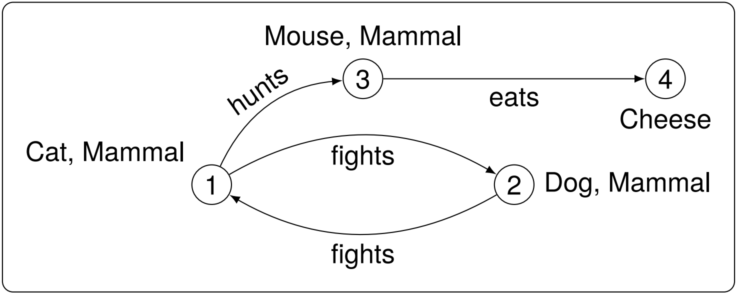
```

Approach

Unstructured data is given as a *finite interpretation* (finite vertex- and edge-labeled graphs)

A Simple Example

Example (Interpretation $\mathcal{I}_{\text{pets}}$)



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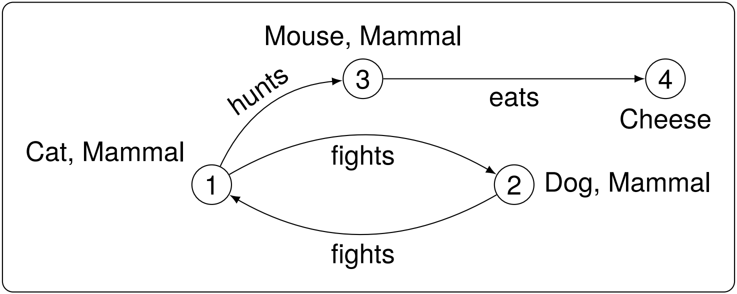

The *elements* (vertices) satisfying $C = \text{Mammal} \sqcap \exists \text{hunts.Mouse}$ are

$$C^{\mathcal{I}} = \{ 1 \}.$$

$C^{\mathcal{I}}$ is called the *extension* of C.

Terminological Knowledge of Interpretations

Goal

Extract all *terminological knowledge*, i. e. all valid GCIs, from $\mathcal{I}$.

Terminological Knowledge of Interpretations

Definition

Let C, D be $\mathcal{EL}^{\perp}$-concept descriptions. Then the GCI $C \sqsubseteq D$ *holds* in $\mathcal{I}$ if and only if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.

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Cat $\sqsubseteq$ Mammal holds in $\mathcal{I}_{\text{pets}}$,

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$\mathsf{Cat} \sqsubseteq \mathsf{Mammal}$ holds in $\mathcal{I}_{\mathsf{pets}}$, and so do

$$\exists\mathsf{hunts}.\mathsf{Cat} \sqsubseteq \exists\mathsf{hunts}.\mathsf{Mammal},$$
$$\exists\mathsf{hunts}.\exists\mathsf{hunts}.\mathsf{Cat} \sqsubseteq \exists\mathsf{hunts}.\exists\mathsf{hunts}.\mathsf{Mammal}, \ldots$$

Bases of Valid GCIs

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Consider *bases* of valid GCIs of $\mathcal{I}$, i. e. sets $\mathcal{B}$ of GCIs such that

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Find a *finite* base of all valid GCIs of $\mathcal{I}$.

Theorem (Baader, Distel 2008)

Finite bases of all valid $\mathcal{EL}^{\bot}$-GCIs of $\mathcal{I}$ always exists. One can be constructed effectively.

Problem: Errors in DBpedia

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Approach assumes data set $\mathcal{I}$ to be complete and free of errors.

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The GCI

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Idea

Consider *confident GCIs*, i. e. GCIs that allow some few “exceptions.”

Formal Concept Analysis

From Mathematical Order Theory to a «Theory of Data»

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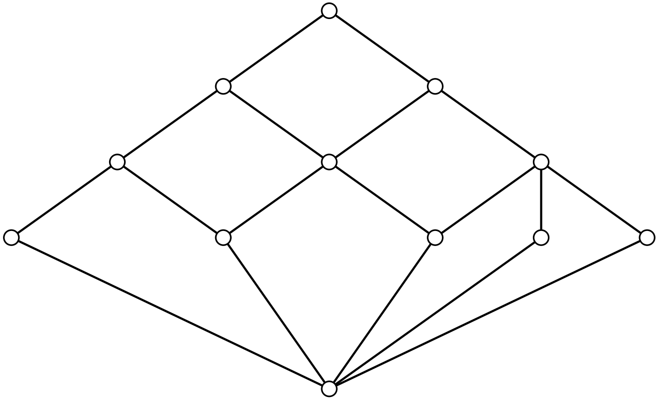

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Literature

- Restructuring Lattice Theory: An Approach Based on Hierarchies of Concepts; R. Wille 1982
- Formal Concept Analysis Mathematical Foundations; R. Wille and B. Ganter; 1999

Formal Contexts

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Uhm …

Meaning?

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Let $\mathbb{K} = (G, M, I)$ be a formal context.
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Let $\mathbb{K} = (G, M, I)$ be a formal context.
We then introduce the following interpretation:

- Elements of G are called *objects* (Gegenstände)
- Elements of M are called *attributes* (Merkmale)
- We say that the object g *has* the attribute m if and only if $(g, m) \in I$

Formal Contexts – Graphical Representation

	1	2	3	4	5
1	×	×	×	×	×
2		×	×	×	×
3			×	×	×
4				×	×
5					×

Formal Contexts – Graphical Representation

	size			distance from sun		moon	
	small	medium	large	near	far	yes	no
Mercury	×			×			×
Venus	×			×			×
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Neptune		×			×	×	
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Formal Concepts

Definition (Derivation Operators)

Let $A \subseteq G, B \subseteq M$. Then we define

$$A' := \{\, m \in M \mid \forall g \in A \colon g\ I\ m \,\},$$
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The set of all formal contexts of $\mathbb{K}$ is denoted by $\mathfrak{B}(\mathbb{K})$.

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Mercury	×			×			×
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Example (Formal Concepts)

- $(\{\text{Mercury, Venus, Earth, Mars, Pluto}\}, \{\text{small}\})$

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Pluto	×				×	×	

Example (Formal Concepts)

- ({ Mercury, Venus, Earth, Mars, Pluto }, { small }) $\hat{=}$ *small planets*
- ({ Pluto }, { small, far, moon }) $\hat{=}$ *small planets far away from sun*

Concept Lattices

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Concepts can be ordered by *generality*.

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Definition

Let $(A_1, B_1), (A_2, B_2) \in \mathfrak{B}(\mathbb{K})$. Then define

$$(A_1, B_1) \leqslant (A_2, B_2) \iff A_1 \subseteq A_2.$$

Concept Lattices

	size			distance from sun		moon	
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Mercury	×			×			×
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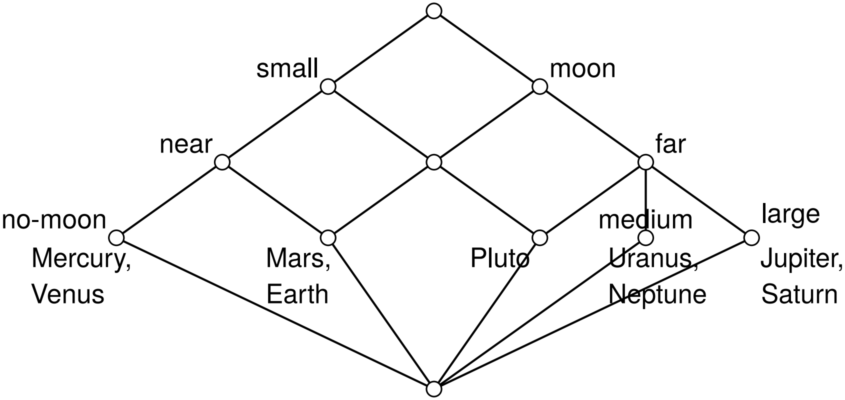

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Observation

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$$\text{far planet} \mathrel{\hat{=}} (\{\,\text{Jupiter, Saturn, Uranus, Neptune, Pluto}\,\}, \{\,\text{far, moon}\,\}).$$

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Definition (Implication (Syntax))

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This is a *model-based* semantics!

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Want to find a finite base of all GCIs of a finite interpretation

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In terms of FCA

Find all valid implications of $\mathbb{K}$

Bases of Implications

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Want to find a finite base of all GCIs of a finite interpretation

In terms of FCA

Find *a good representation* of all valid implications of $\mathbb{K}$

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$\mathcal{B}$ is called a *base* if it is sound and complete. $\mathcal{B}$ is called an *irredundant* base if $\mathcal{B}$ is a base and every proper subset $\mathcal{B}' \subsetneq \mathcal{B}$ is not a base.

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One can explicitly describe some bases of $\mathbb{K}$

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Remark

One can explicitly describe a base of $\mathbb{K}$ *with minimal cardinality*, the so-called *canonical base* of $\mathbb{K}$.

Description Logics

Formalizing Knowledge The Right Way

What are Description Logics about?

In a Nutshell

Description Logics are formal languages to represent knowledge that provide methods to reason about this knowledge.

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The Plan

- Syntax of $\mathcal{ALC}$
- Semantics of $\mathcal{ALC}$
- TBoxes, ABoxes and Ontologies
- Standard Reasoning Tasks

Literature

Baader et. al. ; *The Description Logic Handbook, Theory, Implementation and Applications*; Cambridge University Press; second edition; 2007

Syntax of $\mathcal{ALC}$

Fix the following sets:

- N_C of *concept names*
- N_R of *role names*

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Example

$$N_C = \{ \text{Person}, \text{Male}, \text{Female} \}$$
$$N_R = \{ \text{hasChild} \}$$

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- $C \sqcap D$ for $C, D \in \mathcal{C}$ (conjunction)
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- $\forall r.C$ for $r \in N_R, C \in \mathcal{C}$ (value restriction)
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Example

$$\text{Person} \sqcap \text{Female} \sqcap \exists \text{hasChild}.\top \sqcap \forall \text{hasChild}.\text{Male}$$

A mother which has only sons.

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Interpretations

Example

Consider $\Delta_{\mathcal{I}} = \{ 1, 2, 3, 4 \}$ and

$$\begin{aligned} \mathsf{Person}^{\mathcal{I}} &= \{ 1, 2, 3, 4 \} \\ \mathsf{Male}^{\mathcal{I}} &= \{ 2, 3 \} \\ \mathsf{Female}^{\mathcal{I}} &= \{ 1, 4 \} \\ \mathsf{hasChild}^{\mathcal{I}} &= \{ (1, 3), (2, 3), (3, 4) \}. \end{aligned}$$

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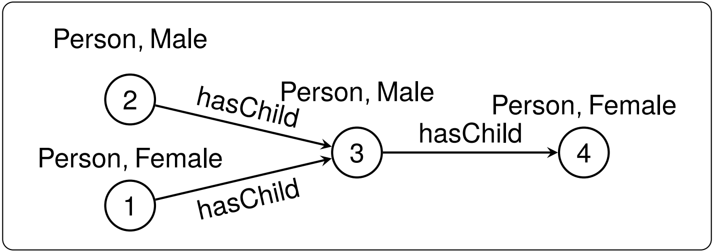

Semantics of $\mathcal{ALC}$

Definition

Let C, D be $\mathcal{ALC}$-concept descriptions, $r \in N_R$.

$$\begin{aligned}
\top^{\mathcal{I}} &= \Delta_{\mathcal{I}} \\
\bot^{\mathcal{I}} &= \emptyset \\
(C \sqcap D)^{\mathcal{I}} &= C^{\mathcal{I}} \cap D^{\mathcal{I}} \\
(C \sqcup D)^{\mathcal{I}} &= C^{\mathcal{I}} \cup D^{\mathcal{I}} \\
(\forall r.C)^{\mathcal{I}} &= \{\, x \in \Delta_{\mathcal{I}} \mid \forall y \in \Delta_{\mathcal{I}} \colon (x, y) \in r^{\mathcal{I}} \implies y \in C^{\mathcal{I}} \,\} \\
(\exists r.C)^{\mathcal{I}} &= \{\, x \in \Delta_{\mathcal{I}} \mid \exists y \in \Delta_{\mathcal{I}} \colon (x, y) \in r^{\mathcal{I}} \wedge y \in C^{\mathcal{I}} \,\}
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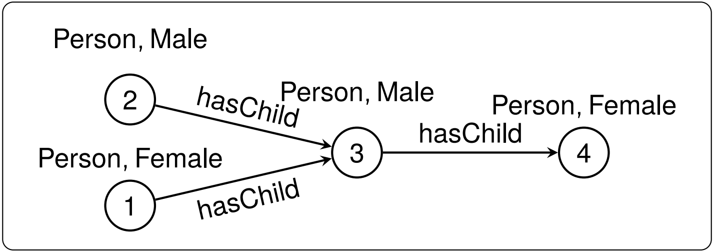

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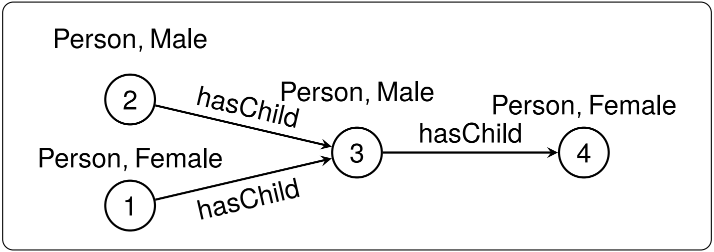

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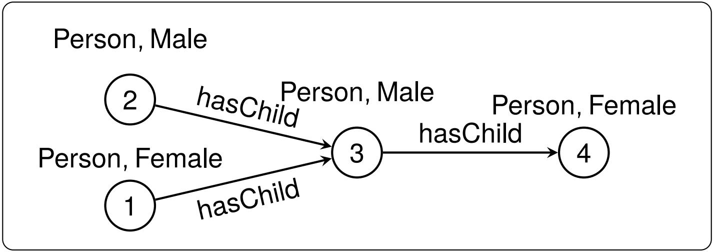

$$(\mathsf{Person} \sqcap \mathsf{Female} \sqcap \exists \mathsf{hasChild}.\top \sqcap \forall \mathsf{hasChild}.\mathsf{Male})^{\mathcal{I}} = \{\, 1 \,\}$$

Description Logic Ontologies

Goal

Use Description Logics to represent knowledge

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- *assertional knowledge*, i. e. “Tom is a cat”
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Definition (Ontology)

An *ontology* is a pair $(\mathcal{T}, \mathcal{A})$, where $\mathcal{T}$ is a TBox and $\mathcal{A}$ is an ABox.

Description Logic Ontologies – An Example

Example ($\mathcal{EL}^{\bot}$-Ontology)

$(\mathcal{T}, \mathcal{A})$ is an ontology, where

$$\begin{aligned}\mathcal{T} &= \{\, \mathsf{Cat} \sqsubseteq \mathsf{Animal} \sqcap \exists \mathsf{hunts}.\mathsf{Mouse}, \\ &\qquad \mathsf{Cat} \sqcap \mathsf{Mouse} \sqsubseteq \bot \,\} \\ \mathcal{A} &= \{\, \mathsf{Cat}(\mathsf{Tom}), \mathsf{Mouse}(\mathsf{Jerry}), \mathsf{hunts}(\mathsf{Tom}, \mathsf{Jerry}) \,\}\end{aligned}$$

Terminological Knowledge and TBoxes

Definition (Terminological Axioms)

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Example

$$\mathcal{T} = \{\, \text{Cat} \sqsubseteq \text{Animal} \sqcap \exists \text{hunts.Mouse}, \\ \text{Cat} \sqcap \text{Mouse} \sqsubseteq \bot \,\}$$

TBox Semantics

Definition (Descriptive Semantics)

An interpretation $\mathcal{I} = (\Delta_{\mathcal{I}}, \cdot^{\mathcal{I}})$ is a *model* of a TBox $\mathcal{T}$ if and only if

$$A^{\mathcal{I}} = C^{\mathcal{I}} \quad \text{and} \quad C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$$

for all $(A \equiv C), (C \sqsubseteq D) \in \mathcal{T}$.

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Other semantics:

- greatest fixpoint semantics
- least fixpoint semantics

Confident GCIs of Finite Interpretations

Handling Errors in Knowledge

Work by Baader and Distel

Theorem (Baader, Distel 2008)

Finite bases of all valid $\mathcal{EL}^{\bot}$-GCIs of $\mathcal{I}$ always exists. One can be constructed effectively.

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Plan

- Introduce necessary terminology
- Define *confident GCIs* as an approach to handle errors
- Discuss some relevant ideas from FCA
- Present first results

In More Detail

Theorem

The set

$$\mathcal{B}_2 := \{ \bigsqcap U \sqsubseteq ((\bigsqcap U)^{\mathcal{I}})^{\mathcal{I}} \mid U \subseteq M_{\mathcal{I}} \}$$

is a finite base of $\mathcal{I}$.

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Definition

$$\bigsqcap U := \begin{cases} \top & U = \emptyset \\ \bigsqcap_{V \in U} V & \text{otherwise.} \end{cases}$$

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Let $X \subseteq \Delta_{\mathcal{I}}$. Then $X^{\mathcal{I}}$ denotes the *model-based most-specific concept description* of X in $\mathcal{I}$.

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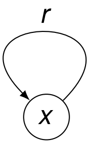
- C is unique up to equivalence, denoted by $X^{\mathcal{I}}$.

Model-Based Most-Specific Concept Descriptions

Problem

Model-based most-specific concept descriptions do not need to exist in $\mathcal{EL}^{\bot}$.



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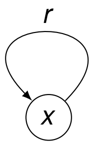

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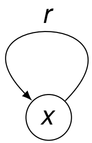

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In $\mathcal{EL}^{\bot}_{\mathrm{gfp}}$ model-based most-specific concept descriptions always exist.

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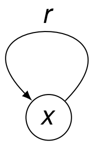

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In $\mathcal{EL}^{\bot}_{gfp}$ model-based most-specific concept descriptions always exist.

Lemma

If $\mathcal{B}$ is an $\mathcal{EL}^{\bot}_{gfp}$-base of $\mathcal{I}$, then one can effectively compute an $\mathcal{EL}^{\bot}$ base $\mathcal{B}'$ from $\mathcal{B}$.

Ontologies from Data: an Example

Experiment (B. 2010)

DBpedia: automatically extracted RDF triples from Wikipedia Infoboxes

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$$\exists \text{hasChild}.\top \sqsubseteq \text{Person}$$

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does not hold in $\mathcal{I}_{\text{DBpedia}}$, but there are only 4 *erroneous* counterexamples.

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DBpedia: automatically extracted RDF triples from Wikipedia Infoboxes

Take relation hasChild $\rightsquigarrow$ interpretation $\mathcal{I}_{\text{DBpedia}}$

$|\Delta_{\mathcal{I}_{\text{DBpedia}}}| = 5626$, Base of GCIs of size 1252.

Observation

$$\exists \text{hasChild}.\top \sqsubseteq \text{Person}$$

does not hold in $\mathcal{I}_{\text{DBpedia}}$, but there are only 4 *erroneous* counterexamples.

Idea

Also consider GCIs that “almost” hold in $\mathcal{I}_{\text{DBpedia}}$.

Confidence of GCIs

Definition

The *confidence* of $C \sqsubseteq D$ in $\mathcal{I}$ is defined as

$$\mathrm{conf}_{\mathcal{I}}(C \sqsubseteq D) := \begin{cases} 1 & \text{if } C^{\mathcal{I}} = \emptyset, \\ \frac{|(C \sqcap D)^{\mathcal{I}}|}{|C^{\mathcal{I}}|} & \text{otherwise.} \end{cases}$$

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Question

Can we find a *finite base* for $\mathrm{Th}_c(\mathcal{I})$?

A Base for Confident GCIs

Answer

There exist finite bases of $\mathsf{Th}_c(\mathcal{I})$.

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Use ideas from Formal Concept Analysis for this!

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For an implication $A \longrightarrow B$ of a formal context $\mathbb{K}$ define its *confidence* to be

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Then: find a set $\mathcal{B} \subseteq \mathrm{Th}_c(\mathbb{K})$ that is *complete* for $\mathrm{Th}_c(\mathbb{K})$, i. e. that entails all implications from $\mathrm{Th}_c(\mathbb{K})$.

Implications with Confidence

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Plan (Luxenburger)

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Lemma

For $A \subseteq B \subseteq C \subseteq M$ it is true that

$$\mathrm{conf}_{\mathbb{K}}(A \longrightarrow C) = \mathrm{conf}_{\mathbb{K}}(A \longrightarrow B) \cdot \mathrm{conf}_{\mathbb{K}}(B \longrightarrow C).$$

Implications with Confidence

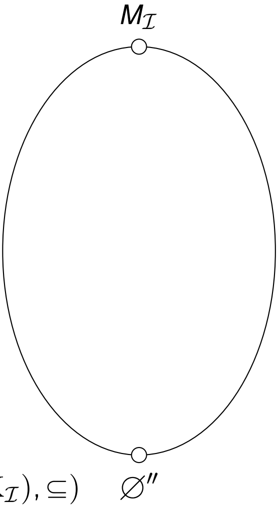
Theorem

Let $\mathbb{K} = (G, M, I)$ *be a finite non-empty formal context and* $c \in [0, 1]$. *Let* $\mathcal{B}$ *be a base of* $\mathbb{K}$ *and define*

$$\mathcal{C} := \{\, A'' \longrightarrow C'' \mid A \subseteq C \subseteq M, \operatorname{conf}_{\mathbb{K}}(A'' \longrightarrow C'') \in [c, 1), \\ \nexists B'' : A'' \subsetneq B'' \subsetneq C'' \,\}.$$

Then $\mathcal{B} \cup \mathcal{C}$ *is a base of* $\operatorname{Th}_c(\mathbb{K})$.

An Order Isomorphism



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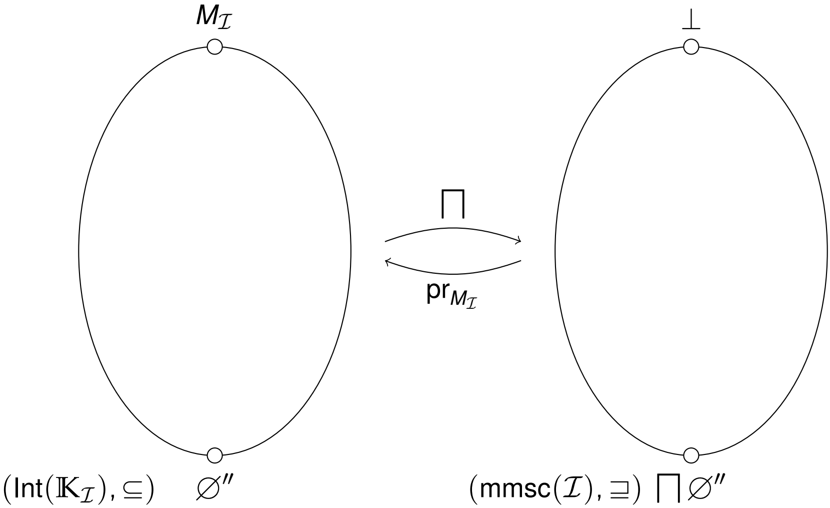

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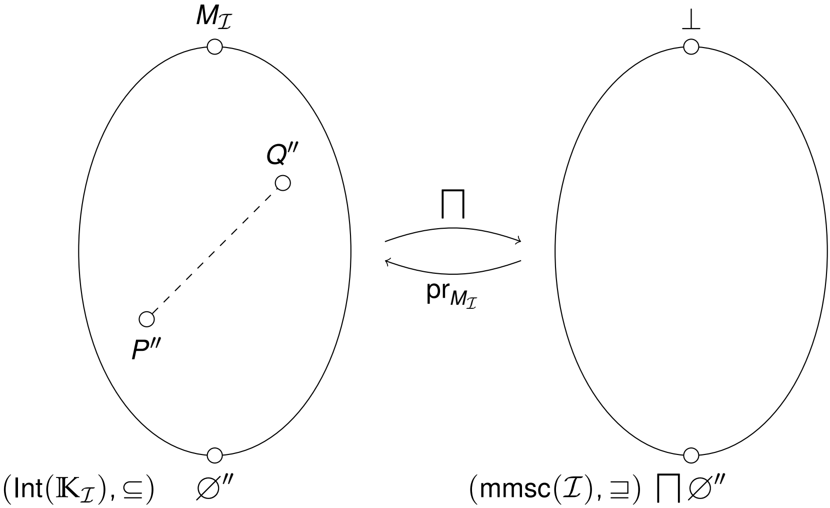

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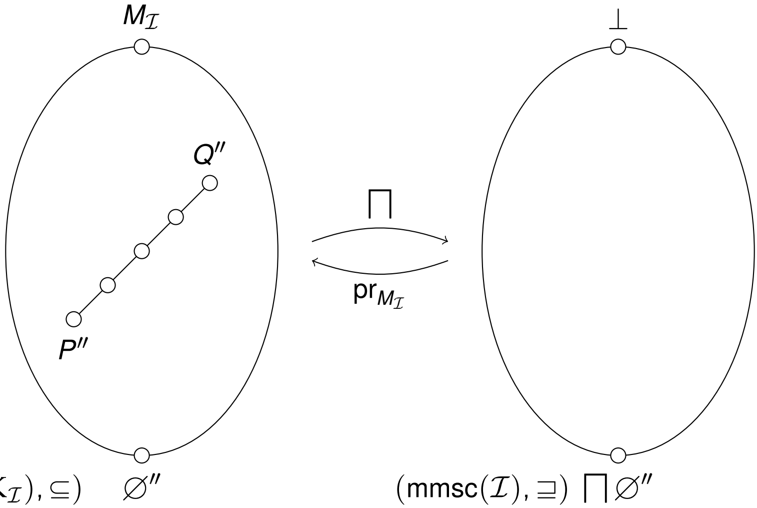

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$(\mathrm{mmsc}(\mathcal{I}), \sqsupseteq)$

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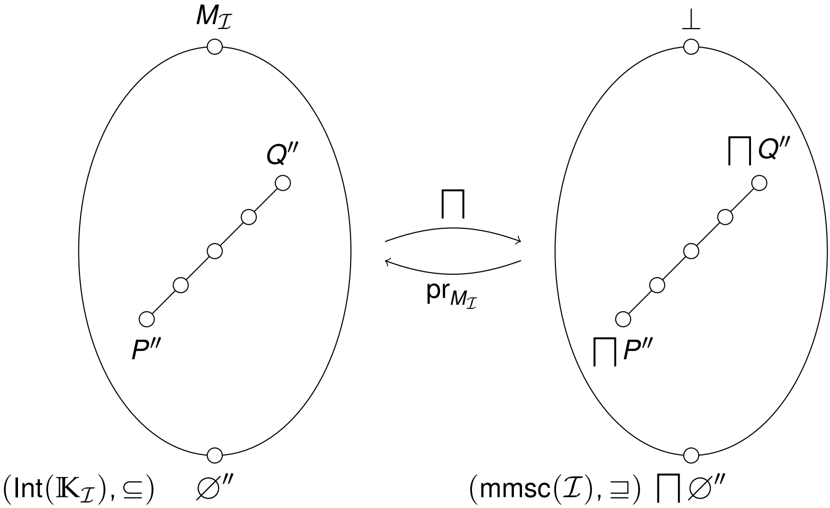

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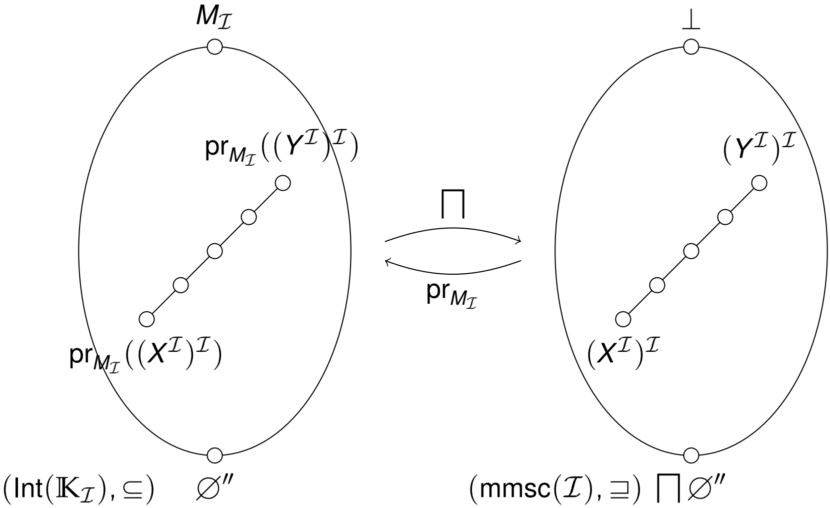

$\operatorname{conf}_{\mathcal{I}}((X^{\mathcal{I}})^{\mathcal{I}} \sqsubseteq (Y^{\mathcal{I}})^{\mathcal{I}}) \in [c, 1)$

A Base for Confident GCIs

Theorem (B. 2012)

Let $\mathcal{B}$ be a finite base of $\mathcal{I}$, $c \in [0, 1]$ and

$$\mathrm{Conf}(\mathcal{I}, c) := \{ X^{\mathcal{I}} \sqsubseteq Y^{\mathcal{I}} \mid Y \subseteq X \subseteq \Delta_{\mathcal{I}}, 1 > \mathrm{conf}_{\mathcal{I}}(X^{\mathcal{I}} \sqsubseteq Y^{\mathcal{I}}) \geqslant c \}.$$

Then $\mathcal{B} \cup \mathcal{C}$ is a finite base of $\mathrm{Th}_c(\mathcal{I})$.

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Theorem (B. 2012)

The set

$$\mathcal{D} := \{\, (X^{\mathcal{I}} \sqsubseteq Y^{\mathcal{I}}) \in \operatorname{Conf}(\mathcal{I}, c) \mid \nexists Z \subseteq \Delta_{\mathcal{I}} \colon Y^{\mathcal{I}} \subsetneq Z^{\mathcal{I}} \subsetneq X^{\mathcal{I}} \,\}$$

is complete for $\mathcal{C}$. In particular, $\mathcal{B} \cup \mathcal{D}$ is a finite base for $\operatorname{Th}_c(\mathcal{I})$.

Thank You for Your Attention!